The Applications Of BCG and PBCG Methods On Stokes Problem Solved By Finite Element Method

Omar Ali Aleyan[1]

Abstract.

We apply the Biconjugate Gradient method (BCG) and preconditioned Biconjugate Gradient method (PBCG) for the system $Hx = b$, that we have got by discretization of the Stokes equation, we show the number of iterations and residual on different mesh size for the two methods, and finally we can compare the results.

Key words. BCG method, PBCG method, Stokes problem, finite element methods.

Introduction.

Let Ω be a bounded domain in R^d, with boundary $\partial\Omega$, and let $f \in L_0^2(\Omega)$ be a given vector function in Ω and $\nu > 0$ is a given constant representing the viscosity of the fluid. We consider the Stokes equation: find u and p where u is a vector of velocity and p is pressure such that

$$
\begin{aligned}
-\nu \nabla^2 u + \nabla p &= f && \text{in } \Omega, \\
\nabla u &= 0 && \text{in } \Omega, \qquad (1)\\
u &= 0 && \text{on } \partial\Omega.
\end{aligned}
$$

We seek $u \in V = [H_0^1(\Omega)]^d$ and $p \in P = L_0^2(\Omega)$, where $L_0^2(\Omega)$ means the space of square integrable functions with mean value zero, $H_0^1(\Omega)$ means the Sobolev space of functions with trace zero.

[1] Mathematics Department, Faculty Of Education, AL Asmarya Islamic University

Finite element discretization of (1) yields linear system of the form $Hx = b$.

(2) $$\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix}\begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix}.$$

Here H is matrix, $A \in R^{n\times n}$, $B \in R^{m\times n}$ and $n \geq m > 0$.
To solve the linear system of the form $Hx = b$. We will apply the two methods BCG and PBCG

The Biconjugate Gradient method (BCG)

To solve the linear system $Hx = b$, by the Biconjugate gradient method (BCG) it takes the following form.

$r_0 = b - Hx_0$, $p_0 = r_0$ and $\tilde{r}_0 = \tilde{b} - H^T\tilde{x}_0$, $\tilde{p}_0 = \tilde{r}_0$

and compute

$$a_j = \frac{\tilde{r}_j^T r_j}{\tilde{p}_j^T H p_j},$$

$$x_{j+1} = x_j + a_j p_j$$

$r_{j+1} = r_j - a_j H p_j$,

$$\tilde{r}_{j+1} = \tilde{r}_j - a_j H^T \tilde{p}_j,$$

$$b_j = \frac{\tilde{r}_{j+1}^T r_{j+1}}{\tilde{r}_j^T r_j},$$

$$p_{j+1} = r_{j+1} + b_j p_j.$$

$$\tilde{p}_{j+1} = \tilde{r}_{j+1} + b_j \tilde{p}_j.$$

$j = 0,1,...$ until $r_{j+1}^T r^T{}_{j+1} \leq \varepsilon$ or $\tilde{r}_{j+1}^T \tilde{r}^T{}_{j+1} \leq \varepsilon$. When near breakdown occurs that is, $\tilde{r}_j^T r^T{}_j$ is close to zero.

The Preconditioned Biconjugate Gradient method (PBCG)

We can use

$r_0 = b - Hx_0$, and $\tilde{r}_0 = \tilde{b} - H^T\tilde{x}_0$, The biconjugate gradient method now makes a special choice and uses the setting

$p_0 = M^{-1} r_0$ and $\tilde{p}_0 = M^{-T} \tilde{r}_0$

and compute

$$a_j = \frac{\tilde{r}_j^T M^{-1} r_j}{\tilde{p}_j^T H p_j},$$

$$x_{j+1} = x_j + a_j p_j$$

$r_{j+1} = r_j - a_j H p_j,$

$$\tilde{r}_{j+1} = \tilde{r}_j - a_j H^T \tilde{p}_j,$$

$$b_j = \frac{\tilde{r}_{j+1}^T M^{-1} r_{j+1}}{\tilde{r}_j^T M^{-1} r_j},$$

$$p_{j+1} = M^{-1} r_{j+1} + b_j p_j .$$

$$\tilde{p}_{j+1} = M^{-T} \tilde{r}_{j+1} + b_j \tilde{p}_j .$$

where P is one of the following preconditioners:

(i) Block Diagonal Preconditioner

Suppose that the matrix H is preconditioned by the preconditioner block diagonal matrix P_{BD}

$P_{BD} = \begin{bmatrix} A & 0 \\ 0 & BA^{-1}B^T \end{bmatrix}$.. The preconditioned matrix is $T_{BD} = P_{BD}^{-1} H$.

(ii) Upper Block Triangular Preconditioner

We consider the matrix H is preconditioned by the preconditioner upper block triangular matrix P_{UBD},

$P_{UBT} = \begin{bmatrix} A & B^T \\ 0 & BA^{-1}B^T \end{bmatrix}$. The preconditioned matrix is $T_{UBT} = P_{UBT}^{-1} H$.

(iii) Lower Block Triangular Preconditioner

Suppose that the matrix H is preconditioned by the preconditioner block triangular matrix P_{IBD}

$P_{LBT} = \begin{bmatrix} A & 0 \\ B & BA^{-1}B^{T} \end{bmatrix}$. The preconditioned matrix is $T_{LBT} = P_{LBT}^{-1}H$.

(iv) The incomplete LU factorization preconditioner

We consider a general sparse matrix H a general Incomplete LU (ILU) factorization process computes a sparse lower triangular matrix L and a sparse upper triangular matrix U. So the residual matrix $R = LU - H$ satisfies certain constraints, such as: having zero entries in some locations.

Application. We will apply BCG and PBCG for this example

Example. As a test problem we study the Stokes problems

$$\nu\nabla^2 u + \nabla p = 0$$
$$\nabla . u = 0$$

With the following parameters $\nu = 0.000025 \quad m^2/s$, velocity max 1 m/s, the boundary conditions on Γ_{IN} are $u_2 = 0$ and $u_1 = 1 - y^2$, on Γ_W zero boundary conditions, on Γ_A $u_1 = 1$ and $u_2 = 0$, on Γ_{OUT} $\nu\frac{\partial u}{\partial n} - pn = 0.$

We show the domain on Figure 1

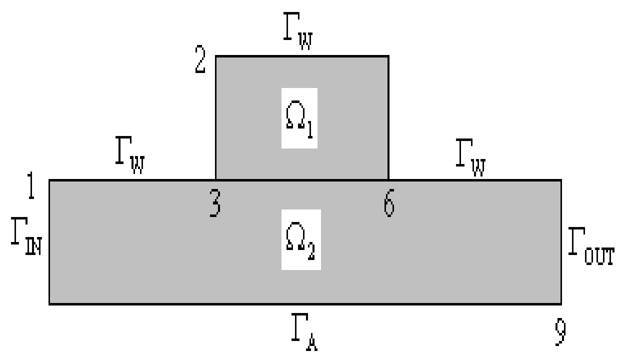

Figure 1 The Domain of the example

Regular mesh of the example

On figure 2 there is mesh of finite element whishes named on the tables as finer mesh the dimensions are $n = 474$, $m = 71$ $n + m = 545$. $H \in R^{545 \times 545}$ is nonsymmetric.(n and m are defined in (1/2). On the figure 3 we show the pattern of matrix H

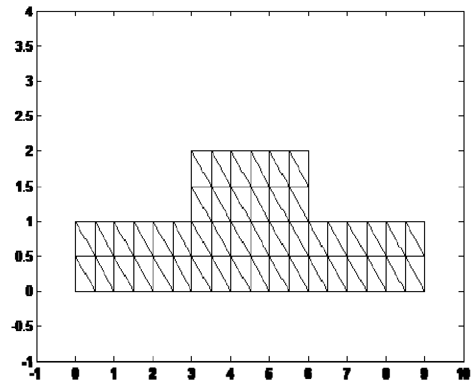

Figure 2 Division into finite elements of the example

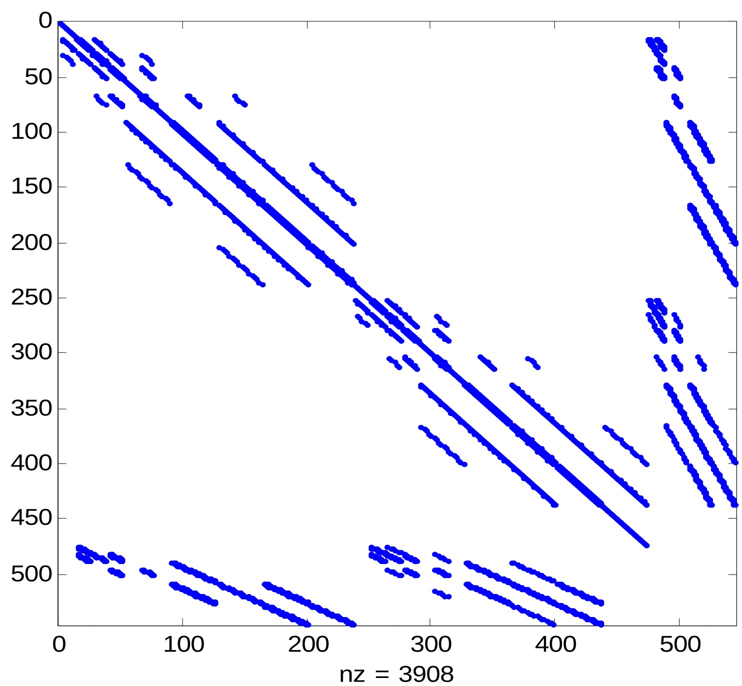

Figure 3 The graph of the matrix H

The Results

Table 1 shows the size f the matrix H of the example

H	1/2	1/3	1/4	1/5
Size of matrix H	545x545	1140x1140	1951x1951	2978x2978

Table 2shows the iteration and residual of matrix H by biconjugate gradient method
of example

h		1/2	1/3	1/4	1/5
BCG	Residual	8.7e-5	8.1e-5	9.7e-5	7e-5
	Iteration	87	120	160	249

Table 3 shows the iteration and residual of matrix H preconditioned by Block Diagonal Preconditioner for the example

h		1/2	1/3	1/4	1/5
T_{BD}	Residual	1.6e-15	1.9e-15	2.2e-15	2.4e-15
	Iteration	3	3	3	3

Table 4shows the iteration and residual of matrix H preconditioned by UpperBlock Triangular Preconditioner for the example

h		1/2	1/3	1/4	1/5
T_{UBD}	Residual	2.1e-15	3.7e-15	2.1e-15	4.7e-15
	Iteration	2	2	2	2

Table 5shows the iteration and residual of matrix H preconditioned by Lower Block Triangular Preconditionerfor the example

h		1/2	1/3	1/4	1/5
T_{LBD}	Residual	3.2e-14	3.3e-14	3.6e-14	6.2e-14
	Iteration	2	2	2	2

Table 6shows the iteration and residual of matrix H preconditioned by the incomplete LU factorization preconditionerfor the example

h		1/2	1/3	1/4	1/5
T_{ILU}	Residual	8.2e-16	1.3e-15	1.5e-15	1.8e-15
	Iteration	1	1	1	1

Conclusion.

It is shown in this paper that in the BCG method the number of iterations increases when the size of matrix H increases. The preconditioned matrix is convergent faster than without using preconditioners. The best preconditioner is incomplete LUfactorization preconditioner.

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