# SOME RESULTS ON THE GOINT NUMERICAL RANGE OF AN N-TUPLE OF OPERATORS ON A SEMI-INNER PRODUCT SPACE

by

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# بعض النتائج في المدى العددي المشترك لمتجه (n-tuple) من المؤثرات على فضاء شبه الجداء الداخلي

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من الحقائق المعروفة في هذا الموضوع هي المبرهنة المتعلقة بالقيم الذاتية (eigen values) للموثر T (حيث $T \in B(H)$، H هو فضاء هلبرت)، والتي تنص على ان :

مجموعة القيم الذاتية $\lambda$ للمؤثر T هي مجموعة جزئية من المدى العددي W(T) للموثر T، وان العكس يكون صحيحا بشرط إن $\|T\| = |\lambda|$ .

إن بعض النتائج التي حصلنا عليها في هذا البحث هي:

- ☒ تعميم المبرهنة أعلاه للحاله العامة n-tuple وليس فقط لفضاء هلبرت فحسب بل لجميع الفضاءات ذات شبه الجداء الداخلي ، مع شرط ضروري للعكس .
- ☒ الطيف النقطوي التقريبي $\pi(T)$ للموثر T يكون مجموعة جزئية من الانغلاق $\overline{W(T)}$ للمدى العددي .
- ☒ تعميم المبرهنة المعروفة والتي تنص على ان $\sigma(T) \subseteq \overline{W(T)}$ $\forall T \in B(H)$ للحالة العامة n-tuple بالنسبة لفضاء هلبرت .
- ☒ تعميم المبرهنة اعلاه في حالة الفضاء شبه الجداء الداخلي .
- ☒ تعميم المبرهنة ( Lumer) والتي تنص على ان :

$$\overline{W(T)} \cap \{\lambda \in \mathbb{C} : |\lambda| = \|T\|\} \subseteq \pi(T), \quad \forall T \in B(H)$$

للحالة العامة n-tuple وبعض النتائج الأخرى .

## Abstract

P.R. Halmos [3] has proved that . for any bounded operator T an a Hilbert space H ,every eigen value $\lambda$ of T is in the numerical range W(T) , and the converse is true if
$|\lambda| = \|T\|$ .

In this paper we generalize the above result to the case of n–tuples of the operators on any normed linear space

(semi-inner product space) with some necessary condition for the converse . Further , we improve a theorem of lumer [5] which asserts that : the approximate point spectrum $\Pi$(T) of $\mathbf{T \in B(X)}$ ,where X is any normed space is contained in

$\overline{\mathbf{W(T)}}$ where dash is denotes closure. We find a necessary condition to have $\boldsymbol{\sigma}\mathbf{(T)} \subseteq \overline{W(T)}$ ,and we discuss theorem of lumer , for normal operators.From our improvement , we will deduce the well-known theorem for $T \in B(H)$, that

$\boldsymbol{\sigma}\mathbf{(T)} \subseteq \overline{W(T)}$ (since W(T) is already convex [5].)

Also we generalize the above theorem [5] to the case of n-tuples of operator on X

Finally , we generalize the theorem [5] , which says that $:\{\lambda \in \mathbf{c} : |\lambda| = \|T\| \cap \} \; \overline{W(T)} \subseteq \Pi(T)$ , $T \in \mathbf{B(X)}$, when sphere of X is uniformly convex , to the case of n-tuples of operators on X .

## Notations

In what follows , H denote a Hilbert space , and T any operator on H (bounded linear transformation $T:H \to H$).

The numerical range W(T) of $T \in \mathbf{B(H)}$ is defined as :

$W(T) = \{ \langle Tx,x \rangle : \|x\| = 1, x \in H \}$

Which it is convex due to toepliz-Hausdorff theorem [2] , we also have $\boldsymbol{\sigma}\mathbf{(T)} \subseteq \overline{\mathbf{W(T)}}$, where $\boldsymbol{\sigma}\mathbf{(T)}$ denotes spectrum of T . The notation $\Pi_o(T)$ denotes the point spectrum of T , which consists of the set point spectrum of T , which defined as :

$$\Pi(T) = \{\lambda \in C : \exists \text{ a seq } \{x_n\} \text{ of unit vectors in } H \text{ such that :}$$
$\|(T-\lambda I)x_n\| \to o$ **as** $n \to \infty$ }= set of approximate eigen values of T .

In 1960 , G. lumer [5] has defined the semi-inner product (s.i.p)on X*X, when X is any complex (real) vector space , as a complex (real) valued function

, [ , ] ; X*X $\rightarrow$ C(or R) satisfying :

[ x+y,z] = [ x,z]+[y,z ], $\forall x,y,z \in X$

$[\alpha x, y] = \alpha[x, y], \forall \alpha \text{ scalor }, x, y \in x$

2- $\text{if } x \neq 0, \text{then} [x.x] > 0$

3-$[x,y]|^2 \leq [x,x].[y,y]$

Lumer [5] has shown that $[x,x] = \|x\|^2$ defined anorm on X , and if H is a Hilbert space then :

[x,y] = <x,y> , $\forall$ x, y $\in$ **H**

i.e < , > is the unique s.i.p. defined on H .

Further , it is proved [5] that every normed linear space can be made into a s. i. p. in infinity many ways .

For $T \in B(X)$, where X is a s.i.p. the numerical range W(T) of T is defined by : W(T) = $\{[Tx.x] : \|x\|=1, x \in X\}$ Which isn't always convex and is not connected [5] .

For an n-tuple $A=(A_1,A_2,\ldots..A_n)$of operatoron H , the joint numerical range W(A) is defined by :

$$W(A) = \{(<A_1x,x>,<A_2x,x>....<A_nx,x>):$$
$$\|x\|=1, x \in X\}$$

which isn't always convex [4]and [1] by taking the counter example that H =$C^2$ and that

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, A=(A_1,A_2) \text{then}$$

$(0,0),(1,0) \in W(A), \text{but} (1/2,0) \notin W(A)$

For an n-tuple $A=(A_1,A_2,\ldots.A_n)$of operators on H ,a point $\lambda=(\lambda_1,\lambda_2,..\lambda_n)$ in $c^n$ is called a joint eigen value $(\lambda\in\Pi(A))$
if $\exists$ a unit vector x in H such that :
$A_i(x)=\lambda_i(x), \qquad i=1,2..n.$

A point $\lambda=(\lambda_1,\lambda_2\ldots.\lambda_n)$ in $c^n$ is called a joint approximate eigen value ($\lambda\in\Pi(A)$)if $\exists$ a seq $(x_k)$of unit vectors in H sugh that :

$\|(A_i-\lambda_i I)\, x_k\|\to 0$ as $k\to\infty$, for $i=1,2,\ldots.n$.

The joint numerical range W(A)of $A=(A_1,A_2,\ldots A_n)$ of operators on X is defined as :

$W(A)=\{([A_1 x,x],[A_2 x,x],\ldots.[A_n x,x]); \|x\|=1, x\in X\}$

Now we will give our main results obtained in this paper in three sections , section 1 gives a generalization for theorem of Halmos concerning eigen values of T on H . In section 2 we improve a theorem lumer [5] with an application for normal operators on X. Also we generalize the above theorem to the case of n-tuples of operators on X . In section 3 we generalize another theorem [5] .

## Main Results

### Section 1:

P.R. Halmos [2] has proved ;the following theorem, for the case of Hilbert spaces .

### Theorem A [2]

for any $T\in B(H)$ , we have

1) $\Pi_o(T)\subseteq W(T)$ .

2)if $\lambda\in W(T)$ with $|\lambda| =\|T\|$ then $\lambda\in\Pi_o(T)$ .

We have proved the analogous case for part (1)of the above theorem for any n-tuple of operators on a s.i.p.s. X. Also we prove the analogous case for part (2) of the above theorem for n-tuples under the necessary condition that : when Schwartz inequality become an equation , the vectors will be linearly dependent .

**Theorem 1 :**

Let X be a s.i.p.s , & let $A=(A_1,A_2, \ldots A_n)$be an n-tuple of operator on X then :

1).. $\Pi_o(A) \subseteq W(A)$ .

2)..if $\lambda=(\lambda_1,\lambda\ \ ,..\lambda_n)$ & $|\lambda_i|=\|A_i\|$, with the property that

$|[x,y]|=\|x\|.\|y\| \rightarrow x\&y$ are linearly dependent . then:

$$\lambda \in \Pi_o(A).$$

**Proof**

suppose that $\lambda=(\lambda_1,\lambda_2,..\lambda_n)$is a joint eigen value of A. Hence $\exists$ **x** in sphere of X , such that :

$A_i(x)=\lambda_i(x)$. for i=1,2,….n

Now ,for each i=1,2,..n we have

$\lambda_i=\lambda_i.\|x\|^2=\lambda_i.[x,x]=[\lambda_i.x,x]=[A_i(x),x]$

i.e $\lambda \in W(A)$. and therefore $\Pi_o(A) \subseteq W(A)$

(2)suppose that $\lambda=(\lambda_1,\lambda_2,..\lambda_n) \in W(A)$ such that $|\lambda_i| =\|A_i\|$and X has the property that :

$|[x.y]|=\|x\|.\|y\| \rightarrow x$ & y are linearly dependent .

Now , since $x \in W(A),\exists$a unit vector x in Xsuch that :

$\lambda_i=[A_ix,x]$. i=1,2,..n

$\|A_i\|=|\lambda_i|=|[A_ix,\ x]| \leq \|A_i(x)\|.\|x\| \leq \|A_i\|.\|x\|^2=\|A_i\|$

$|[A_i(x),\ x]=\|A_i(x)\|.\|x\|$

therefore $A_i(x)$ and x are linearly dependent.

i.e $A_i(x)=\beta_i(x)$. for some scalar $\beta_i$ :i=1,2,..n

Now ,we have :

$B_i=\beta_i[x,\ x]=[\beta_i(x),\ x]=[A_i(x),x]=\lambda_i$, i=1,2,..n

Thus we conclude $\lambda$ is a joint eigen value of the n-tuple A, which completes the proof of the theorem .

**Section 2:**

In this section , we will give theorem B of lumer [5] . we will discuss this theorem , for normal operators and improve it by proving theorem 2 which says that the spectrum $\sigma(T)$ is contained in the convex hull conh $\overline{W(T)}$ of the closure of W(T), while in theorem 3 we find a necessary condition for the implication that $\sigma(T) \subseteq \overline{W(T)}$.The improvement happen because $\Pi(T) \subseteq \sigma(T)$ (see theorem 31.1 in [3]).

From theorem 3 .We conclude a well-known fact about the Hilbert space operators [2] that $\sigma(T) \subseteq \overline{W(T)}, T \in B(H)$ as an easy corollary by using the Teoplitz-Hausdorff theorem for the convexity of W(T).

**Theorem B: [5]**

Let T be any bounded operator on s.i.p. X ,then

$\sigma(T) \subseteq \overline{W(T)}$

In particular $\partial(\sigma(T)) \subseteq \overline{W(T)}$, where $\partial$ denotes boundary, and $\sigma(T)$ denotes spectrum.

**Corollary**

If T is a normal operator on a s.i.p.s X, then $\sigma(T) \subseteq \overline{W(T)}$

**Proof**

Suppose that T is a normal operator , according to theorem 31.1 [3] ,we have $\sigma(T) = \Pi(T)$

From theorem B , we get $\sigma(T) \subseteq \overline{W(T)}$

**Theorem 2**

Let T be any operator on a s.i.p. X, then :

$\sigma(T) \subseteq \mathrm{Conh}(\overline{W(T)})$ where conh denotes the convex hull .

**Proof :**

Suppose that $\boldsymbol{\lambda \in \sigma(T)}$ and suppose that the line L passing through $\boldsymbol{\lambda}$ intersect $\boldsymbol{\delta(\sigma(T))}$ at $\boldsymbol{\lambda_1}$ **and**$\boldsymbol{\lambda_2}$. Applying theorem 66.B[5] we get that $\boldsymbol{\lambda_1}$ **and**$\boldsymbol{\lambda_2}$ are in $\boldsymbol{\Pi(T)}$

According to theorem B (above) we get that $\boldsymbol{\lambda_1}$ **and**$\boldsymbol{\lambda_2}$ are in $\overline{\mathbf{W(T)}}$ and therefore in Conh $\overline{\mathbf{W(T)}}$.

But $\boldsymbol{\lambda} \in$Line segment Joining $\boldsymbol{\lambda_1}$ **and**$\boldsymbol{\lambda_2}$, with the fact that Conh $\overline{\mathbf{W(T)}}$ is convex , we get : $\boldsymbol{\lambda} \in$ Conh $\overline{\mathbf{W(T)}}$, which competes the proof .

**Theorem 3 :**

Let T be any operator on $\boldsymbol{\alpha}$ s.i.p.s X ; if $\boldsymbol{\Pi(T)}$ is convex then $\boldsymbol{\sigma(T)} \subseteq \overline{\mathbf{W(T)}}$.

**Proof :**

From theorem 3 , we have : $\boldsymbol{\sigma(T)} \subseteq \mathbf{Conh(\overline{W(T)})}$.......(1)

But $\overline{\mathbf{W(T)}}$ is convex , & Conh $\overline{\mathbf{W(T)}}$ is the smallest convex set containing $\overline{\mathbf{W(T)}}$, therefore we will get

$$\overline{\mathbf{W(T)}} = \text{Conh } \overline{\mathbf{W(T)}} \quad ........(2)$$

From (1)&(2) we get : $\boldsymbol{\sigma(T)} \subseteq \overline{\mathbf{W(T)}}$ ,and hence the theorem is proved .

**Corollary :**

Let T be any operator on H , then $\boldsymbol{\sigma(T)} \subseteq \overline{\mathbf{W(T)}}$

**Proof :**

By using Theorem 2 we get that

$\boldsymbol{\sigma(T)} \subseteq \mathbf{Conh(\overline{W(T)})}$.......(1) Now according to Teopliz-

Hausdorff theorem we get that $\mathbf{W(T)}$ is convex ,and it can be easily shown that $\overline{\mathbf{W(T)}}$ is also convex .

i.e $\overline{\mathbf{W(T)}}$ = Conh $\overline{\mathbf{W(T)}}$ ……..(2)

From (1)&(2)we get $\boldsymbol{\sigma}\mathbf{(T)} \subseteq \overline{\mathbf{W(T)}}$ which completes the proof.

**Theorem 4**

Let X be a s.i.p.s. and $A=(A_1,A_2,...A_n)$be an n-tuple of operator on X , then

$$\boldsymbol{\Pi}\mathbf{(A)} \subseteq \overline{\mathbf{W(A)}}$$

**Proof:**

Suppose that $\lambda=(\lambda_1,\lambda_2,..\lambda_n) \in \boldsymbol{\Pi}\mathbf{(A)}$ we get $\exists$**aseq**$\{\mathbf{X_K}\}$of unit vectors in x such that :$\|(A_i - \lambda_i I)x_K\| \to 0$ as $k \to \infty$ for i=1,2,3,…n.

$|[A_i(x_k),x_k)-\lambda_i| = |[(A_i-\lambda_i I)x_k,x_k] \le |(A_i-\lambda_i)x_k| \, \| x_k\| = \|(A_i-\lambda_i I)x_k\| \to 0$ as $n \to \infty$

$|[A_i(x_k),x_k]-\lambda_i| \to 0$ as $k \to \infty$

$|([A_1(x_k),x_k],[A_2(x_k),x_k],\ldots[A_n(x_k),x_k]) - (\lambda_1,\lambda_2,..\lambda_n)|=$

$=|([A_1(x_k),x_k] -\lambda_1,[A_2(x_k),(x_k)] -\lambda_2,\ldots,[A_n(x_k),x_k)-\lambda_n)|$

$= [\sum_{I=1}^{n} |[A_i(x_k),x_k]-\lambda_i|^2 ]^{1/2}$

But , $|[A_i(x_k), x_k]-\lambda_i| \to 0$ as $k \to \infty$. i=1,2,..n

Concluding

$$[\sum_{I=1}^{n} |[A_i(x_k),x_k]-\lambda_i|^2 ]^{1/2} \to 0 \text{ as } k \to \infty$$

This means that ,as $k \to \infty$ , we have

$|([A_1(x_k),x_k],[A_2(x_k),x_k],\ldots[A_n(x_k),x_k]) - (\lambda_1,\lambda_2,..\lambda_n)| \dashrightarrow 0$

Thus $\{([A_i(x_k),x_k] ,[A_2(x_k), x_k],\ldots,[A_n(x_k),x_k])\} \to \boldsymbol{\lambda}$ as $k \to \infty$

Thus $(\lambda_1,\lambda_2,\ldots\lambda_n) \in \overline{\mathbf{W(A)}}$.

**Section 3**

In this section we will give below theorem C of Lumer [5]. We generalize this result to the case of n- tuples of operators on a semi-inner product space .

**Theorem C[5]**

Let X be a s.i.p.s such that the unit sphere in the induced norm is uniformly convex. Let T be an operator on X. Then

$$\{\lambda \in \mathbf{C} : |\lambda| = \|\mathbf{T}\|\} \cap \overline{\mathbf{W(T)}} \subseteq \partial(\sigma(\mathbf{T})) \subseteq \Pi(\mathbf{T})$$

**Theorem 5**

Let X be a s.i.p.s such that the unit sphere in the induced norm is uniformly convex . Let A =($A_1$, $A_2$,…$A_n$) be an n–tuple of operators on X ,Then :

$$\{\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_n)\in c^n : |\lambda_i| = \|A_i\|\} \cap \overline{\mathbf{W(T)}} \subseteq \Pi(\mathbf{A})$$

**Proof :**

Suppose without loss of generality that :

$\|A_i\|=1$,and $\lambda_i=1$

i.e $\lambda=(1,1,\ldots 1)$

Since $\lambda\in W(A)$, $\exists$ a sequence $\{x_k\}$of unit vector in X such that :

$$\{ ([A_1(x_k),x_k)],[A_2(x_k), x_k)],\ldots.[(A_n(x_k),x_k) ]) \}\to 0 \text{ as } k\to\infty$$

Now $\left([\frac{X_K + A_1(X_K)}{2},x_k],[\frac{X_K + A_2(X_K)}{2}, x_k],\ldots.[\frac{X_K + A_n(X_K)}{2},x_k]\right)=$

$=1/2[x_k,x_k],[x_k,x_k],\ldots[x_k,x_k])+1/2([A_1(x_k),x_k],\ldots[A_n(x_k),x_k])$

$=1/2(1,1,\ldots.,1)+1/2([A_1(x_k),x_k],[A_2(x_k),x_k],\ldots[A_n(x_k),x_k])$

$\xrightarrow{\text{CONVERAES TO}} 1/2(1,1,\ldots 1)+1/2(1,1,\ldots 1)=(1,1,\ldots.,1)=\lambda$

Therefore we get

$\left\{\left([\frac{\mathbf{x_K + A_1(x_K)}}{\mathbf{2}},x_k],[\frac{\mathbf{x_K + A_2(x_K)}}{\mathbf{2}}, x_k],\ldots.[\frac{\mathbf{x_K + A_n(x_K)}}{\mathbf{2}},x_k]\right)\right\}$ converges to the point $\lambda=(1,1,\ldots.1)$

i.e $\left|\left([\frac{\mathbf{x_K + A_1(x_K)}}{\mathbf{2}},x_k],[\frac{\mathbf{x_K + A_2(x_K)}}{\mathbf{2}}, x_k],\ldots.[\frac{\mathbf{x_K + A_n(x_K)}}{\mathbf{2}},x_k]\right)-(1,1\ldots,1)\right|$ converge to 0,

i.e as $k\to\infty$ we get

$$0 \le \mathrm{I}\left[\frac{x_K + A_i(x_K)}{2}, x_k\right] - 1\mathrm{I} \le \left(\sum_{I=1}^{N} \frac{x_K + A_i(x_K)}{2}, x_K\right] - 1\mathrm{I}^2\Big)^{1/2}$$

Thus ,for i=1,2,…n as $k\to\infty$ we get

$$\mathrm{I}\left[\frac{x_K + A_i(x_K)}{2}, x_k\right] - 1\mathrm{I} \to 0$$

i.e as $k\to\infty$ , we have $\left[\frac{x_K + A_i(x_K)}{2}, x_k\right] - 1 \to 0$

Therefore ,for i=1,2,…,n we get

$$\left[\frac{x_K + A_i(x_K)}{2}, x_k\right] \to 1 \text{ as } k\to\infty$$

But

$$\left[\frac{x_K + A_i(x_K)}{2}, x_k\right] \le \mathrm{I}\left[\frac{x_K + A_i(x_K)}{2}, x_k\right] \le \mathrm{II}\frac{x_K + A_i(x_K)}{2}\mathrm{II}.\mathrm{I}x_k\mathrm{II}$$

$$\le 1/2([\mathrm{II}x_k\mathrm{II} + \mathrm{II}A_i\,\mathrm{II}.\ \mathrm{II}x_K\,\mathrm{II}).1) = 1/2(1+1) = 1$$

i.e $\left[\frac{x_K + A_i(x_K)}{2}, x_k\right] \le \mathrm{II}\frac{x_K + A_i(x_K)}{2}, x_k\,\mathrm{II} \le 1$

But , $\left[\frac{x_K + A_i(x_K)}{2}, x_k\right] \to 1$ as $k\to\infty$

Thus $\mathrm{II}\frac{X_K + A_i(X_K)}{2}\mathrm{II} \to 1$ as $k\to\infty$

But the unit sphere of X is uniformly convex ,

$\mathrm{II}x_k - A_i(x_k)\,\mathrm{II} \to 0$ as $k\to\infty$

i.e $\mathrm{II}\,(I-A_i)x_K\,\mathrm{II} \to 0$ as $k\to\infty$

i.e (1,1,…,1) is in $\overline{\mathbf{W(A)}}$

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