

Some Properties for Generalized Sakaguchi Type Functions of Complex Order

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Abstract

In this paper we shall study some properties of spirallike and Robertson functions of order α . Also, subordination results and integral means for generalized Sakaguchi type functions of complex order.

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1. Introduction

Let A denote the class of the functions of the form :

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad (1.1)$$

which are analytic and univalent in the open unit disc $U = \{z : |z| < 1\}$. Let $f \in A$ be given by (1.1) and $g(z)$ be given by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k, \quad (1.2)$$

and let T denote the subclass of A whose elements can be expressed in the form:

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k \quad (a_k \geq 0) \quad (1.3)$$

Definition 1 (Hadamard product or convolution). Let a function f defined by (1.1) and g defined by (1.2), the Hadamard product (or convolution)

($f * g$) is defined by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z).$$

Definition 2 [9]. A function $f(z) \in A$ is in $S^{\lambda}(\alpha)$, the class of λ -spirallike functions of order α ($0 \leq \alpha < 1, |\lambda| < \frac{\pi}{2}$), if and only if

$$\operatorname{Re} \left\{ e^{i\lambda} \frac{zf'(z)}{f(z)} \right\} > \alpha \cos \lambda \quad (z \in U). \quad (1.4)$$

We note that : $S^{\lambda}(0) = S^{\lambda}$, the class of λ -spirallike functions was introduced by Spacek [25] and $S^0(0) = S^*$ (see Silverman [19]). Further, a function $f(z) \in A$ is said to be in the class $C^{\lambda}(\alpha)$, the class of λ -Robertson functions of order α , if and only if

$$\operatorname{Re} \left\{ e^{i\lambda} \left(1 + \frac{zf''(z)}{f'(z)} \right) \right\} > \alpha \cos \lambda \quad (z \in U). \quad (1.5)$$

We note that $C^{\lambda}(0) = C^{\lambda}$, the class of functions $f(z)$, for which $zf'(z)$ is λ -spirallike in U introduced by Robertson [17] and the class $C^{\lambda}(\alpha)$ was introduced and studied by Chichra [3] and Sizuk [24] and $C^0(\alpha) = K(\alpha)$ were first introduced by Robertson [17].

We note that :

$$f(z) \in C^{\lambda}(\alpha) \Leftrightarrow zf'(z) \in S^{\lambda}(\alpha).$$

We denote by K the class of all convex functions (see Silverman [19]).

Also, Let Ω denote to the class of bounded analytic functions $\omega(z)$ in U and satisfying the conditions: $\omega(0) = 0$ and $|\omega(z)| < |z|$ for $z \in U$.

Definition 3 [14] (Subordination Principle). For two functions $f(z)$ and $F(z)$, analytic in U , we say that $f(z)$ is subordinate to $F(z)$, written symbolically as follows:

$$f \prec F \text{ in } U \text{ or } f(z) \prec F(z) \quad (z \in U),$$

if there exists a Schwarz function $\omega(z) \in \Omega$, which (by definition) is analytic in U with

$$\omega(0) = 0 \text{ and } |\omega(z)| < 1 \quad (z \in U)$$

such that

$$f(z) = F(\omega(z)) \quad (z \in U).$$

Indeed it is known that

$$f(z) \prec F(z) (z \in U) \Rightarrow f(0) = F(0) \text{ and } f(U) \subset F(U).$$

In particular, if the function $F(z)$ is univalent in U , we have the following equivalence

$$f(z) \prec F(z) (z \in U) \Leftrightarrow f(0) = F(0) \text{ and } f(U) \subset F(U).$$

An analytic function $f(z) \in A$ is said to be in the generalized Sakaguchi class $S(\alpha, s, t)$, if it satisfies

$$\operatorname{Re} \left\{ \frac{(s-t)zf'(z)}{f(sz) - f(tz)} \right\} > \alpha \quad (z \in U) \quad (1.6)$$

for some $\alpha (0 \leq \alpha < 1)$, $s, t \in \mathbb{C}, |t| \leq 1, s \neq t$ and for $z \in U$. The class $S(\alpha, s, t)$ defined and studied by Frasin [4] (see also [18], [11], [12]).

Remark 1. Putting $s = 1$, we obtain $S(\alpha, 1, t) = \Sigma(\alpha, t)$ studied by Owa et al. [15] and Goyal and Goswami [6].

For A, B fixed $-1 \leq B < A \leq 1, s, t \in \mathbb{C}, |t| \leq 1, s \neq t$ and $b \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$, we define the subclass $S(s, t; b; A, B)$ of A consisting of functions f of the form (1.1) as follows:

$$1 + \frac{1}{b} \left(\frac{(s-t)zf'(z)}{f(sz) - f(tz)} - 1 \right) \prec \frac{1 + Az}{1 + Bz}. \quad (1.7)$$

From (1.7) and the definition of subordination, we obtain

$$1 + \frac{1}{b} \left(\frac{(s-t)zf'(z)}{f(sz) - f(tz)} - 1 \right) = \frac{1 + A\omega(z)}{1 + B\omega(z)} \quad (\omega(z) \in \Omega)$$

and hence we have

$$\left| \frac{\frac{(s-t)zf'(z)}{f(sz) - f(tz)} - 1}{B \frac{(s-t)zf'(z)}{f(sz) - f(tz)} - [B + (A - B)b]} \right| < 1. \quad (1.8)$$

We let $T(s, t; b; A, B) = S(s, t; b; A, B) \cap T$, where T is defined by (1.3).

We note that for suitable choices of s, t, b, A and B , we obtain the following

subclasses:

(i) Putting $s = b = 1, t = 0, A = 1$ and $B = -1$, we have $S(1, 0, 1; 1, -1) = S^*$.

(ii) Putting $s = b = 1, t = 0, A = 1 - 2\alpha (0 \leq \alpha < 1)$ and $B = -1$, we have

$$S(1, 0, 1; 1 - 2\alpha, -1) = S^*(\alpha) \text{ (see Robertson [16]);}$$

(iii) Putting $s = b = 1, t = 0, A = (1 - 2\alpha)\beta$ and $B = -\beta (0 \leq \alpha < 1, 0 < \beta \leq 1)$, we have

$$S(1, 0, 1; (1 - 2\alpha)\beta, -\beta) = S(\alpha, \beta) \text{ (see Gupta and Jain [7]);}$$

(iv) Putting $s = 1, t = 0$ and $b = e^{-i\lambda} \cos \lambda (|\lambda| < \frac{\pi}{2})$, we have

$$S(1, 0, e^{-i\lambda} \cos \lambda; A, B) = S^\lambda(A, B) \text{ (see Aouf [1], with } \alpha = 0 \text{);}$$

(v) Putting $s = 1, t = 0, b = e^{-i\lambda} \cos \lambda (|\lambda| < \frac{\pi}{2}), A = 1$ and $B = -1$, we have

$$S(1, 0, e^{-i\lambda} \cos \lambda; 1, -1) = S^\lambda \text{ (see Spacek [25]);}$$

(vi) Putting $s = 1, t = 0, b = e^{-i\lambda} \cos \lambda (|\lambda| < \frac{\pi}{2}), A = 1 - 2\alpha (0 \leq \alpha < 1)$ and $B = -1$, we have

$$S(1, 0, e^{-i\lambda} \cos \lambda; 1 - 2\alpha, -1) = S^\lambda(\alpha) \text{ (see Libera [9]);}$$

(vii) Putting $s = 1, t = 0, b = e^{-i\lambda} \cos \lambda (|\lambda| < \frac{\pi}{2}), A = (1 - 2\alpha)\beta$ and

$$B = -\beta (0 \leq \alpha < 1, 0 < \beta \leq 1), \text{ we have}$$

$$S^\lambda(1, 0, e^{-i\lambda} \cos \lambda; (1 - 2\alpha)\beta, -\beta) = S^\lambda(\alpha, \beta);$$

(viii) Putting $b = (1 - \alpha)e^{-i\lambda} \cos \lambda (|\lambda| < \frac{\pi}{2}), A = 1$ and $B = -1$, we have

$$S(s, t, (1 - \alpha)e^{-i\lambda} \cos \lambda; 1, -1) = \Re^\lambda(\alpha, s, t) \text{ (see Mathur et al. [13]).}$$

(ix) Putting $b = e^{-i\lambda} \cos \lambda (|\lambda| < \frac{\pi}{2})$, we have $S(s, t, e^{-i\lambda} \cos \lambda; A, B) = \Re^\lambda(s, t, A, B)$.

Definition 4 [27]. A sequence $\{c_k\}_{k=1}^\infty$ of complex numbers is said to be

subordinating factor sequence if, whenever $f(z)$ of the form (1.1) is analytic, univalent and convex in U and

$$\sum_{k=1}^{\infty} a_k c_k z^k \prec f(z) \quad (z \in U; a_1 = 1). \quad (1.9)$$

2. Main Results

First, we shall obtain some subordination results.

Unless otherwise mentioned we assume throughout the paper that $-1 \leq B < A \leq 1, s, t, \in \mathbb{C}, |t| \leq 1, s \neq t, b \in \mathbb{C}^*$.

To prove our main results we need the following lemmas.

Lemma 1 [27]. The sequence $\{c_k\}_{k=1}^\infty$ is subordinating factor sequence if and only

if

$$\operatorname{Re}\left\{1+2\sum_{k=1}^{\infty}c_k z^k\right\}>0 \quad (z \in U).$$

Now, we prove the following lemma which gives a sufficient condition for functions belonging to the class $S(s,t;b;A,B)$.

Lemma 2. A function $f(z)$ of the form (1.1) is in the class $S(s,t;b;A,B)$ if

$$\sum_{k=2}^{\infty}[|k-u_k|(1-B)+(A-B)|b||u_k||a_k|]\leq(A-B)|b|, \quad (2.1)$$

Where
$$u_k = \sum_{j=1}^k s^{k-j} t^{j-1}.$$

Proof. From (1.8), we have

$$\left|\frac{(s-t)zf'(z)}{f(sz)-f(tz)}-1\right|-\left|B\frac{(s-t)zf'(z)}{f(sz)-f(tz)}-\left[B+(A-B)b\right]\right|<0,$$

and

$$\frac{\sum_{k=2}^{\infty}|k-u_k||a_k|-B\sum_{k=2}^{\infty}|k-u_k||a_k|}{1-\sum_{k=2}^{\infty}|u_k||a_k|}\leq(A-B)|b|$$

and hence

$$\sum_{k=2}^{\infty}[|k-u_k|(1-B)+(A-B)|b||u_k||a_k|]\leq(A-B)|b|.$$

Lemma 3. A necessary and sufficient condition for $f(z) \in T$ to be in the class $T(s,t;b;A,B)$ is that

$$\sum_{k=2}^{\infty}[|k-u_k|(1-B)+(A-B)|b||u_k||a_k|]\leq(A-B)|b|.$$

Proof. In view of Lemma 1, we need to prove the necessarily

$$\left|\frac{\frac{(s-t)zf'(z)}{f(sz)-f(tz)}-1}{B\frac{(s-t)zf'(z)}{f(sz)-f(tz)}-\left[B+(A-B)b\right]}\right|$$

$$= \left| \frac{-\sum_{k=2}^{\infty} k u_k a_k z^k + \sum_{k=2}^{\infty} u_k a_k z^k}{Bz - B \sum_{k=2}^{\infty} k u_k a_k z^k - [B + (A - B)b](z - \sum_{k=2}^{\infty} u_k a_k z^k)} \right|$$

$$= \left| \frac{-\sum_{k=2}^{\infty} (k - u_k) a_k z^k}{-(A - B)bz + (A - B)b \sum_{k=2}^{\infty} u_k a_k z^k - B \sum_{k=2}^{\infty} (k - u_k) a_k z^k} \right| < 1 \quad (z \in U).$$

Since $\operatorname{Re}(z) \leq |z|$ for all z , we have

$$\operatorname{Re} \left\{ \frac{\sum_{k=2}^{\infty} (k - u_k) a_k z^k}{(A - B)bz - (A - B)b \sum_{k=2}^{\infty} u_k a_k z^k + B \sum_{k=2}^{\infty} (k - u_k) a_k z^k} \right\} < 1.$$

Choose values of z on the real axis so that $\frac{(s-t)zf'(z)}{f(sz)-f(tz)}$ is real. Then upon clearing the denominator in the last result and letting $z \rightarrow 1^-$ through real values, we have

$$\sum_{k=2}^{\infty} [k - u_k] [(1 - B) + (A - B)b |u_k|] |a_k| \leq (A - B)b$$

This completes the proof of Lemma 3.

Remark 2. (i) Putting $b = (1 - \alpha)e^{i\lambda} \cos \lambda$ ($0 \leq \alpha < 1, |\lambda| < \frac{\pi}{2}$), $A = 1$ and $B = -1$ in Lemma

2, we obtain the result obtained by Mathur et al. [13, Theorem 2.3];

(ii) Putting $s = 1, t = 0, b = e^{i\lambda} \cos \lambda$ ($|\lambda| < \frac{\pi}{2}$), $A = 1$ and $B = -1$ in Lemma 2, we obtain the result obtained by Silverman [20] (see also Singh [23])

Let us denote by $S^*(s, t; b; A, B)$, the class of functions $f(z) \in A$ whose coefficients satisfy the inequality (2.1). We note that $S^*(s, t; b; A, B) \subset S(s, t; b; A, B)$.

Employing the technique used earlier by Attiya [2] and Srivastava and Attiya [26], we prove:

Theorem 1. Let $f(z) \in A$ satisfy the inequality (2.1). Then for every $h \in \kappa$, we have

$$\frac{|2-s-t|(1-B)+(A-B)|b||s+t|}{2[|2-s-t|(1-B)+(A-B)|b||s+t|+(A-B)|b|]}(f * h)(z) < h(z) (z \in U), \quad (2.2)$$

and

$$\operatorname{Re}(f(z)) > -\frac{|2-s-t|(1-B)+(A-B)|b||s+t|+(A-B)|b|}{|2-s-t|(1-B)+(A-B)|b||s+t|}, \quad (z \in U). \quad (2.3)$$

The constant factor $\frac{|2-s-t|(1-B)+(A-B)|b||s+t|}{2[|2-s-t|(1-B)+(A-B)|b||s+t|+(A-B)|b|]}$ in the subordination result (2.2)

is the best possible.

Proof. Let $f(z) \in S^*(s, t; b; A, B)$ and let $h(z) = z + \sum_{k=2}^{\infty} d_k z^k \in k$. Then we have

$$\begin{aligned} & \frac{|2-s-t|(1-B)+(A-B)|b||s+t|}{2[|2-s-t|(1-B)+(A-B)|b||s+t|+(A-B)|b|]}(f * h)(z) = \\ & \frac{|2-s-t|(1-B)+(A-B)|b||s+t|}{2[|2-s-t|(1-B)+(A-B)|b||s+t|+(A-B)|b|]} \left(z + \sum_{k=2}^{\infty} a_k d_k z^k \right). \end{aligned} \quad (2.4)$$

Thus, by Definition 4, the subordination result (2.2) will hold true if the sequence

$$\left\{ \frac{|2-s-t|(1-B)+(A-B)|b||s+t|}{2[|2-s-t|(1-B)+(A-B)|b||s+t|+(A-B)|b|]} a_k \right\}_{k=1}^{\infty} \quad (2.5)$$

is a subordinating factor sequence, with $a_1 = 1$. In view of Lemma 1, this is equivalence to the following inequality:

$$\operatorname{Re} \left\{ 1 + \sum_{k=1}^{\infty} \frac{|2-s-t|(1-B)+(A-B)|b||s+t|}{|2-s-t|(1-B)+(A-B)|b||s+t|+(A-B)|b|} a_k z^k \right\} > 0 (z \in U). \quad (2.6)$$

Now, since

$$\Psi(k) = |k - u_k|(1-B) + (A-B)|b||u_k|$$

is an increasing function of k ($k \geq 2$), we have

$$\operatorname{Re} \left\{ 1 + \sum_{k=1}^{\infty} \frac{|2-s-t|(1-B)+(A-B)|b||s+t|}{|2-s-t|(1-B)+(A-B)|b||s+t|+(A-B)|b|} a_k z^k \right\}$$

$$\begin{aligned}
 &= \operatorname{Re} \left\{ 1 + \frac{|2-s-t|(1-B)+(A-B)|b||s+t|}{|2-s-t|(1-B)+(A-B)|b||s+t|+(A-B)|b|} z \right. \\
 &\quad \left. + \frac{1}{|2-s-t|(1-B)+(A-B)|b||s+t|+(A-B)|b|} \sum_{k=2}^{\infty} \left[|2-s-t|(1-B)+(A-B)|b||s+t| \right] a_k z^k \right\} \\
 &\geq 1 - \frac{|2-s-t|(1-B)+(A-B)|b||s+t|}{|2-s-t|(1-B)+(A-B)|b||s+t|+(A-B)|b|} r \\
 &\quad - \frac{1}{|2-s-t|(1-B)+(A-B)|b||s+t|+(A-B)|b|} \sum_{k=2}^{\infty} \left[|2-s-t|(1-B)+(A-B)|b||s+t| \right] |a_k| r^k \\
 &> 1 - \frac{|2-s-t|(1-B)+(A-B)|b||s+t|}{|2-s-t|(1-B)+(A-B)|b||s+t|+(A-B)|b|} r - \frac{(A-B)|b|}{|2-s-t|(1-B)+(A-B)|b||s+t|+(A-B)|b|} r \\
 &> 0 \quad (|z|=r < 1),
 \end{aligned}$$

where we have also made use of assertion (2.1) of Lemma 2. Thus (2.6) holds true in U . This proves the inequality (2.2). The inequality (2.3) follows from (2.2) by taking the convex function $h(z) = \frac{z}{1-z} = z + \sum_{k=2}^{\infty} z^k$.

To prove the sharpness of the constant $\frac{|2-s-t|(1-B)+(A-B)|b||s+t|}{2[|2-s-t|(1-B)+(A-B)|b||s+t|+(A-B)|b|]}$, we consider the function $f_0(z) \in S^*(s, t; b; A, B)$ is given by

$$f_0(z) = z - \frac{(A-B)|b|}{|2-s-t|(1-B)+(A-B)|b||s+t|} z^2. \quad (2.7)$$

Thus from the relation (2.2), we obtain

$$\frac{|2-s-t|(1-B)+(A-B)|b||s+t|}{2[|2-s-t|(1-B)+(A-B)|b||s+t|+(A-B)|b|]} (f_0)(z) < \frac{z}{1-z} \quad (z \in U). \quad (2.8)$$

It can easily be verified that

$$\min_{|z| \leq 1} \operatorname{Re} \left\{ \frac{|2-s-t|(1-B)+(A-B)|b||s+t|}{2[|2-s-t|(1-B)+(A-B)|b||s+t|+(A-B)|b|]} (f_0)(z) \right\} = -\frac{1}{2}. \quad (2.9)$$

This shows that the constant $\frac{|2-s-t|(1-B)+(A-B)|b||s+t|}{2[|2-s-t|(1-B)+(A-B)|b||s+t|+(A-B)|b|]}$ is the best possible. This completes the proof of Theorem 1.

Remark 3. (i) Putting $b = (1-\alpha)e^{i\lambda} \cos \lambda$ ($0 \leq \alpha < 1, |\lambda| < \frac{\pi}{2}$), $A=1$ and $B=-1$ in

Theorem 1, we obtain the result obtained by Mathur et al. [13, Theorem 3.1];

(ii) Putting $s=1, t=0$, $b=e^{i\lambda} \cos \lambda$ ($|\lambda| < \frac{\pi}{2}$), $A=1$ and $B=-1$ in Theorem 1, we obtain

the result obtained by Singh [23, Theorem 2.1];

(iii) Putting $s=1, t=0, b=(1-\alpha)e^{i\lambda}\cos\lambda (0\leq\alpha<1, |\lambda|<\frac{\pi}{2}), A=1$ and $B=-1$ in

Theorem 1, we obtain the result obtained by Kwon and Owa [8, Theorem 2.4];

(iv) Putting $s=b=1, t=0, A=1$ and $B=-1$ in Theorem 1, we obtain the result obtained by Frasin [5, Corollary 2.4];

(v) Putting $s=b=1, t=0, A=1-2\alpha (0\leq\alpha<1)$ and $B=-1$ in Theorem 1, we obtain the result obtained by Frasin [5, Corollary 2.3].

Also, we establish subordination results for the associated subclasses,

$$S^*(\alpha, \beta), S^{*\lambda}(A, B), S^{*\lambda}(\alpha, \beta), \Re^{*\lambda}(s, t, A, B).$$

Putting $s=b=1, t=0, A=(1-2\alpha)\beta$ and $B=-\beta (0\leq\alpha<1, 0<\beta\leq1)$ in Theorem 1, we obtain the following corollary

Corollary 1. Let the function $f(z)$ defined by (1.1) be in the class $S^*(\alpha, \beta)$ and suppose that $h(z) \in \mathbf{\kappa}$. Then

$$\frac{1+\beta(3-2\alpha)}{2[1+\beta(5-4\alpha)]}(f * h)(z) \prec h(z) (z \in U), \quad (2.10)$$

and

$$\operatorname{Re}(f(z)) > -\frac{1+\beta(5-4\alpha)}{1+\beta(3-2\alpha)}, \quad (z \in U).$$

The constant factor $\frac{1+\beta(3-2\alpha)}{2[1+\beta(5-4\alpha)]}$ in the subordination result (2.10) is the best possible.

Putting $s=1, t=0$ and $b=e^{-i\lambda}\cos\lambda$ in Theorem 1, we obtain the following corollary.

Corollary 2. Let the function $f(z)$ defined by (1.1), be in the class $S^{*\lambda}(A, B)$ and suppose that $h(z) \in \mathbf{\kappa}$. Then

$$\frac{(A-B)+(1-B)\sec\lambda}{2[2(A-B)+(1-B)\sec\lambda]}(f * h)(z) \prec h(z) (z \in U), \quad (2.11)$$

and

$$\operatorname{Re}(f(z)) > -\frac{2(A-B)+(1-B)\sec\lambda}{(A-B)+(1-B)\sec\lambda}, \quad (z \in U).$$

The constant factor $\frac{(A-B)+(1-B)\sec\lambda}{2[2(A-B)+(1-B)\sec\lambda]}$ in the subordination result (2.11) is the

best possible.

Putting $s=1, t=0$, $b=e^{-i\lambda} \cos \lambda, A=(1-2\alpha)\beta$ and $B=-\beta$ ($0 \leq \alpha < 1, 0 < \beta \leq 1$), in Theorem 1, we obtain the following corollary.

Corollary 3. Let the function $f(z)$ defined by (1.1) be in the class $S^{*\lambda}(\alpha, \beta)$ and suppose that $h(z) \in \mathcal{K}$. Then

$$\frac{2\beta(1-\alpha)+(1+\beta)\sec \lambda}{2[4\beta(1-\alpha)+(1+\beta)\sec \lambda]}(f * h)(z) \prec h(z) \quad (z \in U), \quad (2.12)$$

and

$$\operatorname{Re}(f(z)) > -\frac{4\beta(1-\alpha)+(1+\beta)\sec \lambda}{2\beta(1-\alpha)+(1+\beta)\sec \lambda}, \quad (z \in U).$$

The constant factor $\frac{2\beta(1-\alpha)+(1+\beta)\sec \lambda}{2[4\beta(1-\alpha)+(1+\beta)\sec \lambda]}$ in the subordination result (2.12) is the best possible.

Putting $b=e^{-i\lambda} \cos \lambda, A=1$ and $B=-1$ in Theorem 1, we obtain the following corollary.

Corollary 4. Let the function $f(z)$ defined by (1.1) be in the class

$\mathfrak{R}^{*\lambda}(s, t, A, B)$ and suppose that $h(z) \in \mathcal{K}$. Then

$$\frac{|2-s-t|(1-B)+(A-B)\cos \lambda|s+t|}{2[|2-s-t|(1-B)+(A-B)\cos \lambda|s+t|+(A-B)\cos \lambda]}(f * h)(z) \prec h(z) \quad (z \in U), \quad (1.13)$$

and

$$\operatorname{Re}(f(z)) > -\frac{|2-s-t|(1-B)+(A-B)\cos \lambda|s+t|+(A-B)\cos \lambda}{|2-s-t|(1-B)+(A-B)\cos \lambda|s+t|}, \quad (z \in U).$$

The constant factor $\frac{|2-s-t|(1-B)+(A-B)\cos \lambda|s+t|}{2[|2-s-t|(1-B)+(A-B)\cos \lambda|s+t|+(A-B)\cos \lambda]}$ in the subordination result (1.13) is the best possible.

Next, we shall obtain some integral mean results.

In [19], Silverman found that the function $f_2(z) = z - \frac{z^2}{2}$ is often extremal over the family T . He applied this function to resolve his integral means inequality, conjectured in [21] and settled in [22], that

$$\int_0^{2\pi} |f(re^{i\varphi})|^\delta d\varphi \leq \int_0^{2\pi} |f_2(re^{i\varphi})|^\delta d\varphi,$$

for all $f(z) \in T$, $\delta > 0$ and $0 < r < 1$. In [22], Silverman proved his conjecture for the classes $T^*(\alpha)$, where $T^*(\alpha) = S^*(\alpha) \cap T$.

In 1925, Littlewood [10] proved the following subordination theorem.

Lemma 4 [10]. If the functions $f(z)$ and $h(z)$ are analytic in U with $f(z) \prec h(z)$, then for $\delta > 0$ and $z = re^{i\theta}$ ($0 < r < 1$), we have

$$\int_0^{2\pi} |f(re^{i\theta})|^\delta d\theta \leq \int_0^{2\pi} |h(re^{i\theta})|^\delta d\theta.$$

Theorem 2. If $f(z) \in T(s, t, b, A, B)$ and $\delta > 0$, $0 < |z| = r < 1$, then for function $h(z) \in \mathcal{K}$, we have

$$\frac{\varphi(2)}{\varphi(2) + (A - B)|b|} \int_0^{2\pi} |(f * h)(re^{i\theta})|^\delta d\theta \leq 2 \int_0^{2\pi} |h(re^{i\theta})|^\delta d\theta, \quad (2.14)$$

where $\varphi(2) = |2 - s - t|(1 - B) + (A - B)|b||s + t|$.

Proof. The proof follows from Theorem 1 and Lemma 4.

Putting $s = b = 1$, $t = 0$, $A = 1$ and $B = -1$ in Theorem 2, we obtain the following corollary.

Corollary 5. Let the function $f(z)$ defined by (1.3) be in the class S^* and suppose that $h \in \mathcal{K}$. Then

$$\frac{1}{3} \int_0^{2\pi} |(f * h)(re^{i\theta})|^\delta d\theta \leq \int_0^{2\pi} |h(re^{i\theta})|^\delta d\theta, \quad (2.15)$$

Putting $s = b = 1$, $t = 0$, $A = 1 - 2\alpha$ ($0 \leq \alpha < 1$) and $B = -1$ in Theorem 2, we obtain the following corollary.

Corollary 6. Let the function $f(z)$ defined by (1.3) be in the class $S^*(\alpha)$ and suppose that $h \in \mathcal{K}$. Then

$$\frac{2 - \alpha}{2(3 - 2\alpha)} \int_0^{2\pi} |(f * h)(re^{i\theta})|^\delta d\theta \leq \int_0^{2\pi} |h(re^{i\theta})|^\delta d\theta, \quad (2.16)$$

Putting $s = b = 1$, $t = 0$, $A = (1 - 2\alpha)\beta$ and $B = -\beta$ ($0 \leq \alpha < 1, 0 < \beta \leq 1$) in Theorem 2, we obtain the following corollary.

Corollary 7. Let the function $f(z)$ defined by (1.3) be in the class $S^*(\alpha, \beta)$ and suppose that $h \in \mathcal{K}$. Then

$$\frac{1 + \beta(3 - 2\alpha)}{2[1 + \beta(5 - 4\alpha)]} \int_0^{2\pi} |(f * h)(re^{i\theta})|^\delta d\theta \leq \int_0^{2\pi} |h(re^{i\theta})|^\delta d\theta, \quad (2.17)$$

Putting $s=1, t=0$ and $b=e^{-i\lambda} \cos \lambda (|\lambda| < \frac{\pi}{2})$ in Theorem 2, we obtain the following corollary.

Corollary 8. Let the function $f(z)$ defined by (1.3) be in the class $S^{*\lambda}(A, B)$ and suppose that $h \in \mathbf{K}$. Then

$$\frac{(A-B)+(1-B)\sec \lambda}{2(A-B)+(1-B)\sec \lambda} \int_0^{2\pi} |(f * h)(re^{i\theta})|^\delta d\theta \leq 2 \int_0^{2\pi} |h(re^{i\theta})|^\delta d\theta, \quad (2.18)$$

Putting $s=1, t=0$, $b=e^{-i\lambda} \cos \lambda (|\lambda| < \frac{\pi}{2})$, $A=(1-2\alpha)\beta$ and $B=-\beta (0 \leq \alpha < 1, 0 < \beta \leq 1)$ in Theorem 2, we obtain the following corollary.

Corollary 9. Let the function $f(z)$ defined by (1.3) be in the class

$S^{*\lambda}(\alpha, \beta)$ and suppose that $h \in \mathbf{K}$. Then

$$\frac{2\beta(1-\alpha)+(1+\beta)\sec \lambda}{2[4\beta(1-\alpha)+(1+\beta)\sec \lambda]} \int_0^{2\pi} |(f * h)(re^{i\theta})|^\delta d\theta \leq \int_0^{2\pi} |h(re^{i\theta})|^\delta d\theta, \quad (2.19)$$

Putting $b=e^{-i\lambda} \cos \lambda (|\lambda| < \frac{\pi}{2})$, $A=1$ and $B=-1$ in Theorem 2, we obtain the following corollary.

Corollary 10. Let the function $f(z)$ defined by (1.3) be in the class

$\mathfrak{R}^{*\lambda}(s, t, A, B)$ and suppose that $h \in \mathbf{K}$. Then

$$\frac{|2-s-t|(1-B)+(A-B)\cos \lambda|s+t|}{2|2-s-t|(1-B)+(A-B)\cos \lambda|s+t|+(A-B)\cos \lambda} \int_0^{2\pi} |(f * h)(re^{i\theta})|^\delta d\theta \leq \int_0^{2\pi} |h(re^{i\theta})|^\delta d\theta, \quad (2.20)$$

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