AN EFFICIENT PRP-HRM HYBRID CONJUGATE GRADIENT METHOD FOR SOLVING UNCONSTRAINED OPTIMIZATION

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ABSTRACT

The hybrid conjugate gradient methodsare combinations of different conjugate gradient (CG) algorithms to give better performance. This paper develops a new hybrid method of conjugate gradient type,satisfies the sufficient descent condition under the exact line search conditions and becomes globally convergent. Preliminary numerical experiments are tested on a set of unconstrained optimization test problems. The resultsof comparisons showthe computational efficiency of the developed hybrid method by solving selected large-scale benchmark test functions againstsome known algorithmsin the sense of Dolan–More performance profile.

Keywords:Exact line search, Hybrid Conjugate gradient, sufficient descent condition, Global convergence.

1. INTRODUCTION

The conjugate gradient (CG) method is iterative techniques prominently used in solving unconstrained optimization problems due to its good convergence analysis, simplicity, and low memory storage. The unconstrained optimization problem is stated by:

$$min\ f(x): x \in \mathbb{R}^n, \qquad (1)$$

where$x \in \mathbb{R}^n$a real vector with $n \geq 1$and$f(x): \mathbb{R}^n \to \mathbb{R}$is a smooth function,and its gradient$g(x)$is available. A nonlinear conjugate gradient

(CG) method generates a sequence x_k Starting from an initial guess $x_0 \in \mathbb{R}^n$, using the recurrence

$$x_{k+1} = x_k + \alpha_k d_k,\ k=0,1,2,\ldots \quad (2)$$

Where$\alpha_k$is a positive step length which is computed by carrying out a line search, for example, the exact line search where,

$$\alpha_k = \arg m \quad {}_{\alpha\geq 0} f(x_k + \alpha d_k) \qquad (3)$$

The$d_k$is the search direction defined by

$$d_k = \begin{cases} -g_k\,, & f \quad k = 0, \\ -g_k + \beta_k d_{k-1}, & f \quad k \geq 1, \end{cases} \qquad (4)$$

Where$g_k = g(x_k)$ and$\beta_k$is a parameter that determines the different CG methods[1].For instance,

$\beta_k^H = \frac{g_{k+1}^T(g_{k+1}-g_k)}{d_k^T(g_{k+1}-g_k)}, \beta_k^F = \frac{g_k^T g_k}{g_{k-1}^T g_{k-1}}, \beta_k^P = \frac{g_{k+1}^T(g_{k+1}-g_k)}{g_k^T g_k},$
$\beta_k^C = -\frac{g_k^T g_k}{d_{k-1}^T g_{k-1}}, \beta_k^L = -\frac{g_{k+1}^T(g_{k+1}-g_k)}{d_k^T g_k}, \beta_k^D = \frac{\|g_{k+1}\|^2}{d_k^T(g_{k+1}-g_k)},$
$\beta_k^W = \frac{g_k^T\left(g_k - \frac{|g_k|}{|g_{k-1}|} g_{k-1}\right)}{\|g_{k-1}\|^2}, \quad \beta_k^H = \frac{g_k^T\left(g_k - \frac{|g_k|}{|g_{k-1}|} g_{k-1}\right)}{\lambda\|g_{k-1}\|^2 + (1-\lambda)\|d_{k-1}\|^2}, \lambda = 0.4,$ Are theHestenes and Stiefel[2],Fletcher-Reeves[3], Polyak [4],Fletcher [5],Liu and Storey [6],Dai and Yuan [7], Wei, et al. [8], Hamoda, et al. [9]methods respectively.

Many authors have studied the convergence behavior of the above formulas with some line search conditions for many years. They found that some of the classicconjugate gradient methods are theoretically effective and have strong global convergence properties, but they may have modest computational performance due to the jamming problem, while the others may not always be convergent, but they often have better computational (Detailed discussions are available in [10-26]).Accordingly, researchers try to devise some new methods, which have the advantages of both kinds of CG methods. This has been done mostly by combining two or moredifferent CG algorithms in the same CG method to give a better performance. Thus, hybrids try to combine attractive features of different algorithms; for example, Touati-Ahmed and Storey[27] introduced this hybrid CG algorithms,

$$\beta_k^T = \begin{cases} \beta_k^P & i\imath\ 0 \le \beta_k^P \le \beta_k^F, \\ \beta_k^F & o\ he\ \ w \ . \end{cases}$$

Where β_k^T method was a hybrid between the Fletcher-Reeves method andPolak–Ribière–Polyak method[28]. Many researchers developed other common hybrid CG methods such as:

$$\beta_k^H = m \quad = \left\{0, m \quad \{\beta_k^P \quad, \beta_k^F\ \}\right\}$$ (Hu and Story [29])

$$\beta_k^G = m \quad \left\{-\beta_k^F\ , m \quad \{\beta_k^P \quad, \beta_k^F\ \}\right\}$$ (Gilbert and Nocedal[30])

$$\beta_k^{hD} = m \quad \left\{0, m \quad \{\beta_k^H\ , \beta_k^D\ \}\right\}$$ (Dai and Yuan [31])

$\beta_k^C = (1-\theta_k)\beta_k^H + \theta_k\beta_k^D$, and $\theta_k$is a scalar parameter satisfying $0 \le \theta_k \le 1$ (Andrei [32])

For further details on this subject, please refer to these references: [33-37].

The main contribution of this paper is to propose, analyses, and test a hybrid CG method combined with some CG methods to solve unconstrained optimization problems.Therefore,the combinations of the good criteria of *PRP* and*HRM* are used to obtain a better practical hybrid method both in numerical and convergence analysis.In Section 2 of this paper, we describe our proposed hybrid CG method. Analysis of sufficient descent condition with exact line searchand the global convergence of our new methodis given in section 3. Numerical results are reported in section 4. Finally, we end this paper with section 5 where we present the conclusion.

2. NEW HYBRID METHOD

This section introduces a new hybrid CG method, namely *PRP-HRM* method, given by,

$$\beta_k^{PRP-HRM} = \begin{cases} \frac{g_k^T(g_k - g_{k-1})}{\|g_{k-1}\|^2} & \text{if } 0 \le g_k^T g_{k-1} \le \|g_k\|^2 \\ \frac{g_k^T\left(g_k - \frac{\|g_k\|}{\|g_{k-1}\|} g_{k-1}\right)}{0.4\|g_{k-1}\|^2 + 0.6\|d_{k-1}\|^2} & \text{otherwise} \end{cases} \quad (5)$$

where the combination of CG methods used in this paper are known as Polak–Ribière–Polyak method,

$$\beta_k^{PRP} = \frac{g_k^T(g_k - g_{k-1})}{\|g_{k-1}\|^2} [4],$$

andHamoda-Rivaie-Mamatmethod,

$$\beta_k^{HRM} = \frac{g_k^T\left(g_k - \frac{\|g_k\|}{\|g_{k-1}\|} g_{k-1}\right)}{\lambda\|g_{k-1}\|^2 + (1-\lambda)\|d_{k-1}\|^2}, \ \lambda = 0.4 [9, 38].$$

An important feature of $\beta_k^{PRP-HRM}$ is that its value is greater than or equal zero without line search.

From (5) if $0 \le g_k^T g_{k-1} \le \|g_k\|^2$ then,

$$\beta_k^{PRP-HRM} = \frac{g_k^T(g_k - g_{k-1})}{\|g_{k-1}\|^2} = \frac{\|g_k\|^2 - g_k^T g_{k-1}}{\|g_{k-1}\|^2} \ge 0$$

Otherwise,

$$\beta_k^{PRP-HRM} = \frac{g_k^T\left(g_k - \frac{\|g_k\|}{\|g_{k-1}\|} g_{k-1}\right)}{0.4\|g_{k-1}\|^2 + 0.6\|d_{k-1}\|^2} \ge \frac{\|g_k\|^2 - \frac{\|g_k\|}{\|g_{k-1}\|}|g_k^T g_{k-1}|}{0.4\|g_{k-1}\|^2 + 0.6\|d_{k-1}\|^2}.$$

Using the properties of absolute value and Cauchy-Schwartz inequalities imply that,

$$\beta_k^{PRP-HRM} \ge \frac{\|g_k\|^2 - \frac{\|g_k\|}{\|g_{k-1}\|}\|g_k\|\|g_{k-1}\|}{0.4\|g_{k-1}\|^2 + 0.6\|d_{k-1}\|^2} = 0,$$

which implies that,

$$\beta_k^{P\quad -H} = \begin{cases} \dfrac{g_k^T(g_k - g_{k-1})}{\|g_{k-1}\|^2} & i\ 0 \le g_k^T g_{k-1} \le \|g_k\|^2 \\ \dfrac{g_k^T\left(g_k - \frac{\|g_k\|}{\|g_{k-1}\|} g_{k-1}\right)}{0.4\|g_{k-1}\|^2 + 0.6\|d_{k-1}\|^2} & o\ he \end{cases} \ge 0$$

The following is a general algorithm for solving optimization by CG methods

Algorithm 2.1

Step 1: Select the initial point x_0 □ d f and compute: $f_0 = f(x_0)$,and $g_0 = $ □$f(x_0)$. Set $d_0 = -g_0$, $k = 0$.

Step 2: Test a criterion for stopping the iterations, if $\|g_0\| \le \varepsilon$ then stop; otherwise, continue with step 3.

Step 3: Compute α_k by exact line search (3).

Step 4: Let $x_{k+1} = x_k + \alpha_k d_k$, $g_{k+1} = g(x_{k+1})$ if $\|g_{k+1}\| < \varepsilon$ then stop.

Step 5: Compute $\beta_k^{P\quad -H}$ by formula (5),and generated d_{k+1} by (4).

Step 6: Set $k = k + 1$ go to Step 2.

3. GLOBAL CONVERGENCE ANALYSIS

In this section, we study the global convergent properties of $\beta_k^{P\quad -H}$ and begin with the sufficient descent condition.

3.1 Sufficient descent condition

For the sufficient descent condition to hold,

$$g_k^T d_k \le -c\|g_k\|^2, \forall k \ge 0, c > 0 \qquad (6)$$

The following theorem shows that $\beta_k^{P \quad -H}$ with exact line search possess the sufficient descent condition.

Theorem 3.1. Let$\{x_k\}$and$\{d_k\}$ sequences generated by algorithm 2.1,$\beta_k^{P \quad -H}$ given as equations (5),then (6) holds for all$k \geq 0$.

Proof

We proof by induction, that if $k = 0$ then $g_0^T d_0 = g_0^T(-g_0) = -\|g_0\|^2$

Hence, the condition holds; now, we need to prove that:

$$g_k^T d_k \leq -c\|g_k\|^2, \text{for } k \geq 1$$

From (4) we have$d_{k+1} = -g_{k+1} + \beta_{k+1}^{P \quad -H} d_k$

Multiply both sides by g_{k+1}^T

$$g_{k+1}^T d_{k+1} = g_{k+1}^T(-g_{k+1} + \beta_{k+1}^{P \quad -H} d_k) = -\|g_{k+1}\|^2 + \beta_{k+1}^{P \quad -H} g_{k+1}^T d_k. \quad (7)$$

For exact line search, we have$g_{k+1}^T d_k = 0$.Thus $g_{k+1}^T d_{k+1} = -\|g_{k+1}\|^2$,

(See, Gilbert and Nocedal [30]). Hence this condition holds for$k + 1$, where $c = 1 > 0$. Therefore, the sufficient descent condition holds.□

3.2 Global convergence properties

To study the global convergence properties, we need to simplify the$\beta_k^{P \quad -H}$, so that the proof will be more straightforward,

Based on $\beta_k^{P \quad -H}$,$i\imath$ $0 \leq g_k^T g_{k-1} \leq \|g_k\|^2$ then,

$$\beta_k^{P-H} = \frac{g_k^T(g_k - g_{k-1})}{\|g_{k-1}\|^2} = \frac{\|g_k\|^2 - g_k^T g_{k-1}}{\|g_{k-1}\|^2} \leq \frac{\|g_k\|^2}{\|g_{k-1}\|^2}$$

which implies that,

$$\beta_k^{P-H} \leq \frac{\|g_k\|^2}{\|g_{k-1}\|^2}. \qquad (8)$$

Otherwise,

$$\beta_k^{P-H} = \frac{g_k^T\left(g_k - \frac{\|g_k\|}{\|g_{k-1}\|} g_{k-1}\right)}{0.4\|g_{k-1}\|^2 + 0.6\|d_{k-1}\|^2} \leq \frac{\|g_k\|^2 - \frac{\|g_k\|}{\|g_{k-1}\|} g_k^T g_{k-1}}{0.4\|g_{k-1}\|^2},$$

using the properties of absolute value and Cauchy-Schwartz inequalities, then

$$\beta_k^{P-H} \leq \frac{\|g_k\|^2 + \frac{\|g_k\|}{\|g_{k-1}\|} |g_k^T g_{k-1}|}{0.4\|g_{k-1}\|^2} \leq \frac{\|g_k\|^2 + \frac{\|g_k\|}{\|g_{k-1}\|} \|g_k\|\|g_{k-1}\|}{0.4\|g_{k-1}\|^2},$$

which implies that,

$$\beta_k^{P-H} \leq \frac{5\|g_k\|^2}{\|g_{k-1}\|^2}. \qquad (9)$$

To prove the global convergence of this method, we first make the following assumption.

Assumption 3.1

(i) $f(x)$is bounded from below on the level set $\Omega = \{x \in R^n, f(x) \leq f(x_0)\}$, where x_0 is the initial point.

(ii) In some neighborhood $N$of Ω, the objective function is continuously differentiable, and its gradient is Lipschitz continuous, that is there exists a constant $l > 0$such that

$$\|g(x) - g(y)\| \leq l\,\|x - y\|, \forall x, y \in N. \qquad (10)$$

Lemma 3.1 (Zoutendijk lemma)

Suppose Assumption (3.1) holds, let x_k be generated by Algorithm (2.1) and d_k satisfy $g_k^T d_k < 0$ for all k, and α_k is obtained by (3), then we have

$$\sum_{k=0}^{\infty} \frac{\left(g_k^T d_k\right)^2}{\|d_k\|^2} < \infty.$$

The proof of this lemma can be seen from Zoutendijk[39].

Theorem 3.2. Suppose that Assumptions (3.1) holds, the sequence $\{x_k\}$ is generated by Algorithm 2.1, if $\|s_k\| = \|\alpha_k d_k\| \to 0$, while $k \to \infty$, then

$$\lim_{k\to\infty} \inf \|g_k\| = 0. \qquad (11)$$

Proof

Case(1): if $0 \le g_k^T g_{k-1} \le \|g_k\|^2$, then $\beta_k^{P-H} = \frac{g_k^T (g_k - g_{k-1})}{\|g_{k-1}\|^2}$

To prove this theorem, we use contradiction. That is, if theorem 3.2 is not valid, then a constant $c > 0$ exists, such that

$$\|g_k\| \ge c. \qquad (12)$$

From (4), we have $d_k = -g_k + \beta_k d_{k-1}$, now by squaring both sides of the equation, we obtain

$$\|d_k\|^2 = (\beta_k)^2 \|d_{k-1}\|^2 - 2\beta_k g_k^T d_{k-1} + \|g_k\|^2. \qquad (13)$$

Since inexact line search, $g_k^T d_{k-1} = 0$ therefore,

$$\|d_k\|^2 = (\beta_k)^2 \|d_{k-1}\|^2 + \|g_k\|^2 \qquad (14)$$

Substituting (8) into (14), then

$$\|d_k\|^2 \le \|g_k\|^2 + \frac{\|g_k\|^4}{\|g_{k-1}\|^4} \|d_{k-1}\|^2$$

Dividing both sides by $\|g_k\|^4$, then

$$\frac{\|d_k\|^2}{\|g_k\|^4} \le \frac{1}{\|g_k\|^2} + \frac{\|d_{k-1}\|^2}{\|g_{k-1}\|^4} \qquad (15)$$

Utilizing (15) recursively and noting that $\|d_0\|^2 = -g_0^T d_0 = \|g_0\|^2$

$$\frac{\|d_k\|^2}{\|g_k\|^4} \le \sum_{i=0}^{k-1} \frac{1}{\|g_i\|^2}$$

Hence,

$$\frac{\|d_k\|^2}{\|g_k\|^4} \le \frac{k}{c^2}$$

Therefore,

$$\frac{\|g_k\|^4}{\|d_k\|^2} \ge \frac{c^2}{k}$$

This implies,

$$\sum_{k=1}^{\infty} \frac{\|g_k\|^4}{\|d_k\|^2} \ge c^2 \sum_{k=1}^{\infty} \frac{1}{k} = \infty$$

This contradicts the Zoutendijk condition in Lemma 3.1. Therefore, the proof is completed⦿[38]

Case(2): if $\beta_k^{P-H} = \frac{g_k^T\left(g_k - \frac{|g_k|}{|g_{k-1}|} g_{k-1}\right)}{0.4\|g_{k-1}\|^2 + 0.6\|d_{k-1}\|^2}$, then the proof precisely the same as shown in reference [9 theorem 3.2].The proof is completed.

Therefore, the new hybrid formula β_k^{P-H} with the exact line search is globally convergent.

4. NUMERICAL RESULTS AND DISCUSSIONS

In this section, we used32nd test functions considered in [40-42] to find the computational results to analyze the efficiency of β_k^{P-H} , where the dimensions of these problems range from 2 to 10000. We performed a comparison with three CG methods*FR*, *PRP*,and*HRM*.For all the CG methods, we considered the stopping condition to be $\varepsilon = 10^{-6}$, that is, the methods were stopped once the condition $\|g_k\| < 10^{-6}$ was satisfied, or the maximum number of iterations of 1000 was reached[43]. For each of the test functions, we used four initial points, starting from a closer point to the solution and moving on to the one that is furthest from it. A list of functions and the initial points used are shown in table 1, where all the problems are solved by the MATLAB program. We used the exact line search to compute the step size. The CPU processor used was Intel (R) CoreTM i5-6200U (2.30GHz), with RAM 8 GB. In some cases, the computation stopped due to the failure of the line search to find the positive step size, and thus it was considered a failure.Numerical results are compared relative to the CPU time and number of iterations. The performance results are shown in figure1 and figure 2, respectively, using a performance profile tool introduced by Dolan and More [44],which has been used extensively for many years to judge the performance of different methods on a set of test problems.

Table 2. A list of problem functions

No	**Function**	**Dimension**	**Initial points**
1	Three Hump	2	-10, 10, 20, 40
2	Leon function	2	2,5,8,10
3	Treccani	2	5, 10, 20, 50
4	Zettl	2	5, 10, 20, 50
5	Booth	2	10, 25, 50, 100
6	Matyas function	2	1,5,10,15
7	Six Hump	2	-10, 10, -8, 8
8	Extended Wood	4	3,5,20,30
9	Quartic function	4	5,10,15,20
10	Colville function	4	2,4,7,10
11	Perturbed Quadratic	2,4, 10,100,500,1000	1, 3, 5, 10
12	Extended Powell	4,20,100,500,1000	2, 4, 6, 8
13	Quadrtic QF2	2,4,10,100,500,1000	5, 20, 50, 100
14	Diagonal 2	2,4,10,100,500,1000	1, 5, 10, 15
15	Diagonal 4	2,4, 10,100,500,1000	1, 3, 6, 12

No	**Function**	**Dimension**	**Initial points**
16	Extended Quadratic Penalty QP2	2,4,10,100,500,1000	10, 20, 30, 50
17	Extended Himmelblau	10,100,500,1000,10000	50,70, 100, 125
18	Extended Rosenbrock	2,4, 10,100,500,1000,10000	13, 25, 30, 50
19	Shallow	2,4, 10,100,500,1000,10000	10, 25, 50, 70
20	Extended Tridiagonal1	2,4, 10,100,500,1000,10000	6, 12, 17, 20
21	Generlyzed Tridiagonal1	2,4,10,100	7, 10, 13, 21
22	Extended white & Holst	2,4,10,100,500,1000,10000	3, 5, 7, 10
23	Generalized Quartic	2,4,10,100,500,1000,10000	1, 2, 5, 7
24	Generlized Tridiagonal2	2, 4, 10, 100	15,18,20,22
25	Fletcher	4, 10, 100,500,1000,10000	3,5,8,9
26	Extended Denschnb	2,4,10,100,500,1000,10000	8, 13, 30, 50
27	Sum Squares	2,4,10,100,500,1000	1, 3, 7, 10
28	Hager	2,4,10,100	7, 10, 15, 23
29	Raydan1	2,4,10,100	1, 3, 7, 10
30	Extended Penalty	2,4,10,100	80,10, 111, 150
31	Extended Beale	2,4,10,100,500,1000,10000	-1, 3, 7, 10
32	Quadratic QF1	2, 4, 10,100, 500, 1000	3,5,8,10

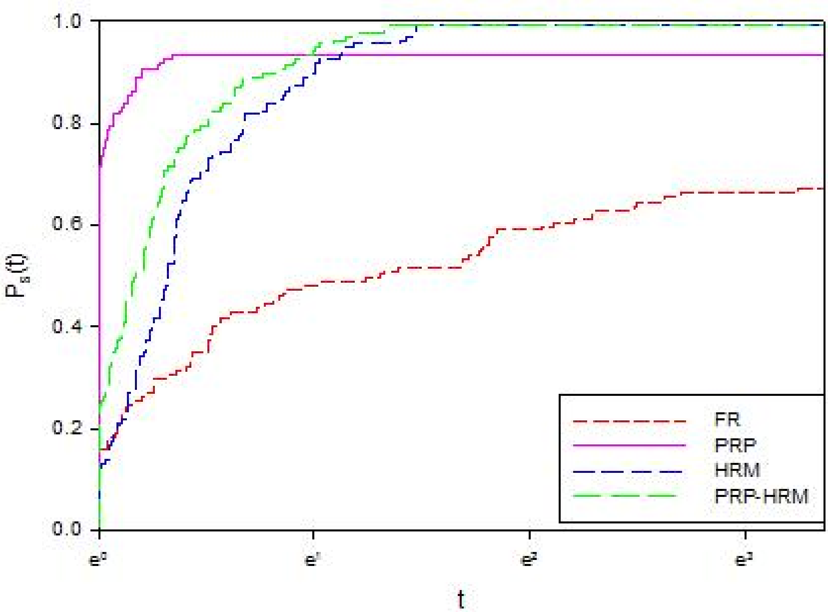

Figure 9: Performance profile relative to the number of iterations

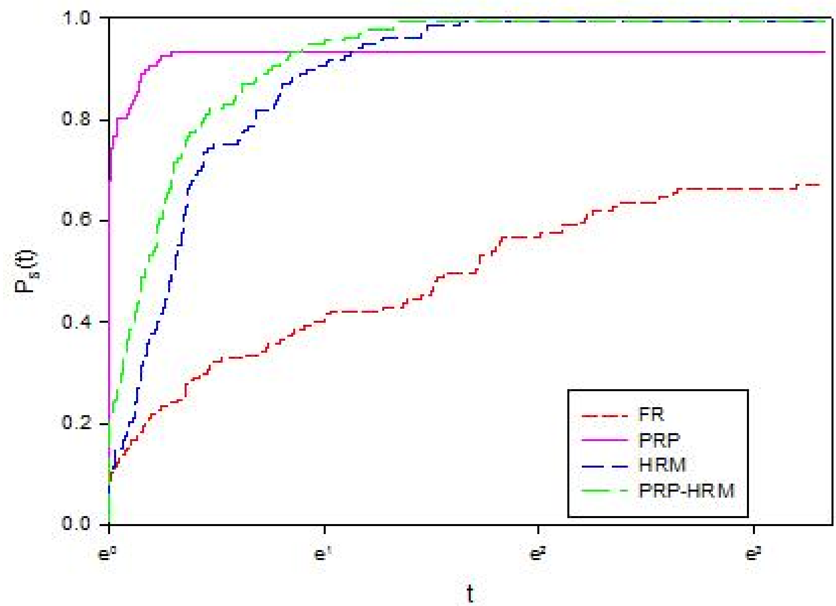

Figure 10: Performance profile relative to the CPU time

From figures 1-2, it is easy to see that the hybrid method*PRP-HRM* is the best among the three methods in the perspectives of the number of iterations and the CPU time. The *PRP-HRM*, *HRM*, and*PRP*methods are much better than*FR*method, where the method*PRP-HRM*is preferable to the*HRM*method. Also, the *HRM*method is preferable to the PRP method, thatis*PRP*can solve 93% of the problems,*HRM*can solve 99% of the problems,and*FR*solved only 67%. Hence, our new hybrid method successfully solved 99% of the test problems. Asfor *PRP-HRM* and *HRM*, we see that in Figures 1 and2, *PRP-HRM* is slightly better than *HRM*,and it is competitive among the well-known conjugate gradient methods for unconstrained optimization.

5. CONCLUSION

In this paper, a hybridization of Polak-Ribiere-Polyak (*PRP*) and Hamoda-Rivaie-Mamat(*HRM*) introduceda new conjugate gradient method named asβ_k^{P-H}, in which the search directions always satisfied the sufficient descent condition. Moreover, an essential property of our proposed method is a global convergence withexact line search. Numerical comparisons have been made between our proposed method and the CG methods (*FR*, *PRP*, and *HRM*) on a set of unconstrained optimization problems. The computational experiments show that our hybrid method is practical, effective, and outperforms the *FR*, *PRP*, and *HRM*.As future work, one can deal with this hybrid method with Armijo type inexact line search for nonconvex unconstrained optimization problems**.**

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