OPTIMUM CONSTANT STRESS ACCELERATED LIFE TESTS FOR THE GENERALIZED LOGISTIC DISTRIBUTION WITH PROGRESSIVE CENSORING

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ABSTRACT
This paper deals with non-Bayesian analysis of constant stress Accelerated Life Test (ALT) when the lifetime of the items follow Generalized Logistic Distribution (GLD). The Maximum Likelihood (ML) method of the parameter estimation and optimal design for Constant Stress Accelerated Life Test (CSALT) under progressive type-I grouped censoring data is contained. It is assumed that the scale parameter be an inverse power law function of the constant stress level. In addition, Fisher information matrix of the estimators are given. Finally, numerical studies are discussed to illustrate the proposed procedure.

Key Words; Accelerated Life Test; Constant stress; Maximum Likelihood; Progressive Censoring; Generalized Logistic Distribution; Inverse Power Law Function; Test Plan.

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1.Introduction

In real life, CSALT is a widely used to assess the reliability of products. Failure information collected under more severe test stress can be extrapolated to obtain an estimate of lifetime under usual conditions based on some life stress relationships. Those stress life relationships are typically used to relate results obtained at stressed conditions to those at usual conditions. Inverse power law relationship has been widely used for accelerated testing involving high pressure stresses (e.g., voltage). Several authors have been used this relationship in CSALT for discussing classical and Bayesian analysis with different lifetime distributions and different censoring schemes. For example, Bai and Chung

(1989) used the ML method to estimate the unknown parameters of constant and progressive stress ALT relationships. There is abundant literature on how to design CSALT. They differ in the assumed lifetime distribution, censoring scheme, and test condition. For example, Ding *et al.* (2010) considered the design of ALT sampling plans under type-I progressive interval censoring with random removals and assumed the lifetime of products follows a Weibull distribution. Attia *et al.* (2011a,b) discussed optimum test plans of the Generalized Logistic (GL) parameters under both type-I and type-II censoring data. It is assumed that the stress affects only the scale GL parameter.

2.Assumptions

We assume the testing assumptions for the CSALT procedure. First, the testing is done at high stresses $V_j,\ j=1,...,k$ where $V_1<V_2< ... <V_k$. Second, let V_u be the design stress; that is the stress level under usual conditions, where $V_u<V_j,\ j=1,2,...,k$. Third, for the testing procedure, a total of N units are divided into $n_1, ... , n_k$ units where $\sum_{j=1}^{k} n_j = N$, each $n_j,\ j=1,\dots,k$ units in the experiment are run at a pre-specified constant stress $V_j,\ j=1,...,k$. Forth, the life of a test unit at each stress is described by three-parameter Generalized Logistic (GL) distribution. Thus, the probability density function (pdf) of the failure times x_{ij} at stress levels $V_j,\ i=1,...,n_j$ and $j=1,...,k$ are

$$f(x_{ij},\alpha_j,\gamma,\theta)=\alpha_j\gamma e^{\alpha_j x_{ij}}(1+\frac{\gamma}{\theta}e^{\alpha_j x_{ij}})^{-(\theta+1)}, -\infty<x_{ij}<\infty,$$
$$\alpha_j,\gamma,\theta>0,\ i=1,...,n_j,\ j=1,...,k. \tag{1}$$

Finally, at the j^{th} stress level, the scale parameter $\alpha_j,\ j=1,\dots,k$ of a test unit is an inverse power law function of stress, that is,

$$\alpha_j=CS_j^P,\ S_j=\frac{V^*}{V_j},\ V^*=\prod_{j=1}^{k}V_j^{b_j},\ b_j=\frac{n_j}{\sum_{j=1}^{k}n_j}, \tag{2}$$

where $C>0$ is the constant of proportionality, and $P>0$ is the power of the applied stress.

3.ML Estimation

Let us consider there are k-stage constant stress ALT scheme with progressive type-I grouped censoring. In this case, the test will be carrying out as follows: n units are simultaneously placed on a life test at stress level V_1, and run until the time x_1. At this time, the number of failed units r_1 are counted and R_1 surviving units are removed from the test; starting from the time x_1, the $(n - r_1 - R_1)$ non-removed surviving units are put to a stress level $V_2, V_1 < V_2$ and run until the time x_2. At this time, the number of failed units r_2 are counted and R_2 surviving units are removed from the test. Starting from the time x_2, the $(n - r_1 - R_1 - r_2 - R_2)$ non-removed surviving units are put to a stress level $V_3, V_2 < V_3$ and run until the time x_3. At this time, the number of failed units r_3 are counted and R_3 surviving units are removed from the test, and so on. At the time x_k, the number of failed units r_k are counted and the remaining surviving units $R_k = n - \sum_{j=1}^{k} r_j - \sum_{j=1}^{k-1} R_j$ are all removed, thereby terminating the test, [Balakrishnan and Aggarwala, 2000].
Let us consider the underlying lifetime distribution as the three-parameter generalized logistic distribution of the test unit at k-stage constant stress level with progressive type-I group censoring. The probability density function (pdf) of the lifetime distribution at the stress level $V_j,\ j = 1, 2, ..., k$ and in the inspection interval (x_{j-1}, x_j) will take the following form
$f(x_j, \alpha_j) = \alpha_j \gamma e^{\alpha_j x_j}(1 + \frac{\gamma}{\theta} e^{\alpha_j x_j})^{-(\theta+1)}, -\infty < x_j < \infty, \alpha_j, \gamma, \theta > 0$, and the cumulative distribution function (cdf) $F(x_j, \alpha_j) = 1 - (1 + \frac{\gamma}{\theta} e^{\alpha_j x_j})^{-\theta}, -\infty < x_j < \infty, \alpha_j, \gamma, \theta > 0,\ j = 1, ..., k.$ where, α_j is defined by equation (2). Under assuming that the experiment is done under k-stage constant stress level, then the number of failed units r_j observed while the testing in the interval (x_{j-1}, x_j) at stress $V_j,\ j = 1, 2, ..., k$ are random variables with the fact that $R_j/r_{j-1}, ..., r_1 \sim Binomial\ [m_j, F_j(x)],\ j = 1, 2, ..., k$, where $m_j = n - \sum_{i=1}^{j-1} r_i - \sum_{i=1}^{j-1} R_i$ is the number of non-removal surviving units at the beginning of the j^{th} stage, and $F_j(x) = \frac{F(x_j) - F(x_{j-1})}{1 - F(x_{j-1})}$. Therefore, the likelihood function can be written as the following form

$$L = \prod_{j=1}^{k} [F(x_j) - F(x_{j-1})]^{r_j} [1 - F(x_j)]^{R_j}. \tag{3}$$

In the special case where the intervals are of equal length, so monitoring and censoring occurs periodically, say $x_j = j \cdot x$ [Aggarwala, 2001]. Therefore, the likelihood function (3) can be re-written as the following form

$$L = \prod_{j=1}^{k} [F(j\ x) - F((j-1)x)]^{r_j} [1 - F(j\ x)]^{R_j}, \tag{4}$$

where

$$F(j\ x, \alpha_j) = 1 - (1 + \frac{\gamma}{\theta} e^{j \alpha_j x})^{-\theta}. \tag{5}$$

According to equations (4) and (5) the likelihood function (3) will be re-written as the following form $L = \prod_{j=1}^{k} [(1 + \frac{\gamma}{\theta} e^{(j-1)CS_j^P x})^{-\theta} - (1 + \frac{\gamma}{\theta} e^{jCS_j^P x})^{-\theta}]^{r_j} (1 + \frac{\gamma}{\theta} e^{jCS_j^P x})^{-\theta R_j}$. The log-likelihood function has the form

$$lnL = \sum_{j=1}^{k} [r_j ln(\psi_{j-1} - \psi_j) + R_j ln\psi_j], \tag{6}$$

where $\psi_{j-1} = (1 + \frac{\gamma}{\theta} e^{(j-1)CS_j^P x})^{-\theta}$, and $\psi_j = (1 + \frac{\gamma}{\theta} e^{jCS_j^P x})^{-\theta}$. The likelihood equations of the log-likelihood function (6) with respect to the unknown parameters C, P, γ, and θ are given by

$\frac{\partial lnL}{\partial C} = \sum_{j=1}^{k} [r_j (\psi_{j-1} - \psi_j)^{-1} (\psi_j \Delta_j - \psi_{j-1} \Delta_{j-1}) - R_j \Delta_j]$,

$\frac{\partial lnL}{\partial P} = C \sum_{j=1}^{k} (lnS_j) [r_j (\psi_{j-1} - \psi_j)^{-1} (\psi_j \Delta_j - \psi_{j-1} \Delta_{j-1}) - R_j \Delta_j]$,

$\frac{\partial lnL}{\partial \gamma} = \frac{\theta}{\gamma} \sum_{j=1}^{k} [r_j (\psi_{j-1} - \psi_j)^{-1} (\psi_j \varphi_j - \psi_{j-1} \varphi_{j-1}) - R_j \varphi_j]$,

$\frac{\partial lnL}{\partial \theta} = \sum_{j=1}^{k} [r_j (\psi_{j-1} - \psi_j)^{-1} [\psi_{j-1} (\varphi_{j-1} - \tau_{j-1}) - \psi_j (\varphi_j - \tau_j)] + R_j (\varphi_j - \tau_j)]$,

where, $\varphi_{j-1} = (1 + \frac{\theta}{\gamma} e^{-(j-1)CS_j^P x})^{-1}$, $\varphi_j = (1 + \frac{\theta}{\gamma} e^{-jCS_j^P x})^{-1}$, $\Delta_{j-1} = (j-1)\ \theta S_j^P x\ \varphi_{j-1}$, $\Delta_j = j\ \theta S_j^P x\ \varphi_j$, $\tau_{j-1} = \frac{-1}{\theta}\ ln\psi_{j-1}$, and $\tau_j = \frac{-1}{\theta}\ ln\psi_j$. The first derivative non-linear equations will be solved numerically as will be seen in Section (5) to obtain the MLE $(\hat{C},\ \hat{P},\ \hat{\gamma},\ \hat{\theta})$. Following, we will obtain the second derivatives in order to obtain Fisher information matrix and the asymptomatic variance-covariance matrix. The second partial derivatives of the log-likelihood function with respect to the parameters $(C,\ P,\ \gamma,\ \theta)$ are as follows

$$\frac{\partial^2 lnL}{\partial C^2} = \sum_{j=1}^{k} [r_j(\psi_{j-1} - \psi_j)^{-1}(\psi_j \Delta_j^2 (1 + \frac{1}{\gamma} e^{-jCS_j^P x}) - \psi_{j-1}\Delta_{j-1}^2 (1 + \frac{1}{\gamma} e^{-(j-1)CS_j^P x}) + (\psi_{j-1} - \psi_j)^{-1} (\psi_j \Delta_j - \psi_{j-1}\Delta_{j-1})^2) - \frac{1}{\gamma} R_j \Delta_j^2 e^{-jCS_j^P x}],$$

$$\frac{\partial^2 lnL}{\partial P^2} = C \sum_{j=1}^{k} (lnS_j)^2 [r_j(\psi_{j-1} - \psi_j)^{-1}(\psi_j \Delta_j (1 + C\Delta_j (\frac{1}{\gamma} e^{-jCS_j^P x} - 1)) - \psi_{j-1}\Delta_{j-1}(1 + C\Delta_{j-1}(\frac{1}{\gamma} e^{-(j-1)CS_j^P x} - 1)) - C(\psi_{j-1} - \psi_j)^{-1}(\psi_j \Delta_j - \psi_{j-1}\Delta_{j-1})^2) - R_j \Delta_j (1 + \frac{C}{\gamma} e^{-jCS_j^P x})],$$

$$\frac{\partial^2 lnL}{\partial \gamma^2} = \frac{\theta}{\gamma^2} \sum_{j=1}^{k} [r_j(\psi_{j-1} - \psi_j)^{-1}(\psi_j \varphi_j (\psi_j^{\frac{1}{\theta}} - \theta\varphi_j) - \psi_{j-1}\varphi_{j-1}(\psi_{j-1}^{\frac{1}{\theta}} - \theta\varphi_{j-1}) - (\psi_j \varphi_j - \psi_{j-1}\varphi_{j-1})(1 + \theta(\psi_{j-1} - \psi_j)^{-1} (\psi_j \varphi_j - \psi_{j-1}\varphi_{j-1}))) + R_j \varphi_j (1 - \psi_j^{\frac{1}{\theta}})],$$

$$\frac{\partial^2 lnL}{\partial \theta^2} = \frac{1}{\theta} \sum_{j=1}^{k} [r_j(\psi_{j-1} - \psi_j)^{-1}(\psi_{j-1}\varphi_{j-1}(1 - \psi_{j-1}^{\frac{1}{\theta}}) - \psi_j \varphi_j (1 - \psi_j^{\frac{1}{\theta}}) + \theta(\psi_{j-1}(\varphi_{j-1} - \omega_{j-1})^2 - \psi_j(\varphi_j - \omega_j)^2) - \theta[(\psi_{j-1} - \psi_j)^{-1} (\psi_{j-1}(\varphi_{j-1} - \omega_{j-1}) - \psi_j(\varphi_j - \omega_j))^2] + R_j \varphi_j (1 - \psi^{\frac{1}{\theta}})],$$

$$\frac{\partial^2 lnL}{\partial C \partial P} = \sum_{j=1}^{k} (lnS_j)[r_j(\psi_{j-1} - \psi_j)^{-1}(\psi_j \Delta_j (1 + C\Delta_j (\frac{1}{\gamma} e^{-jCS_j^P x} - 1)) - \psi_{j-1}\Delta_{j-1}(1 + C\Delta_{j-1}(\frac{1}{\gamma} e^{-(j-1)CS_j^P x} - 1)) - C (\psi_{j-1} - \psi_j)^{-1}(\psi_j \Delta_j - \psi_{j-1}\Delta_{j-1})^2) - R_j \Delta_j (1 + \frac{C}{\gamma} e^{-jCS_j^P x})],$$

$$\frac{\partial^2 lnL}{\partial C \partial \gamma} = \frac{1}{\gamma} \sum_{j=1}^{k} [r_j(\psi_{j-1} - \psi_j)^{-1}(\psi_j \Delta_j (\psi_j^{\frac{1}{\theta}} - \theta\varphi_j) - \psi_{j-1}\Delta_{j-1}(\psi_{j-1}^{\frac{1}{\theta}} - \theta\varphi_{j-1}) - \theta(\psi_{j-1} - \psi_j)^{-1} (\psi_j \Delta_j - \psi_{j-1}\Delta_{j-1})(\psi_j \varphi_j - \psi_{j-1}\varphi_{j-1})) - R_j \Delta_j \psi_j^{\frac{1}{\theta}}],$$

$$\frac{\partial^2 lnL}{\partial C \partial \theta} = \sum_{j=1}^{k} [r_j(\psi_{j-1} - \psi_j)^{-1}(\psi_j \Delta_j ((\varphi_j - \tau_j) + \frac{1}{\theta}(1 - \psi_j^{\frac{1}{\theta}})) - \psi_{j-1}\Delta_{j-1}((\varphi_{j-1} - \tau_{j-1}) + \frac{1}{\theta}(1 - \psi_{j-1}^{\frac{1}{\theta}})) - (\psi_{j-1} - \psi_j)^{-1} (\psi_j \Delta_j - \psi_{j-1}\Delta_{j-1})((\psi_{j-1}(\varphi_{j-1} - \tau_{j-1}) - \psi_j(\varphi_j - \tau_j))) - \frac{1}{\theta} R_j \Delta_j (1 - \psi_j^{\frac{1}{\theta}})],$$

$$\frac{\partial^2 lnL}{\partial P \partial \gamma} = \frac{C}{\gamma} \sum_{j=1}^{k} (lnS_j)[r_j(\psi_{j-1} - \psi_j)^{-1}(\psi_j \Delta_j (\psi_j^{\frac{1}{\theta}} - \theta\varphi_j) - \psi_{j-1}\Delta_{j-1}(\psi_{j-1}^{\frac{1}{\theta}} - \theta\varphi_{j-1}) - \theta(\psi_{j-1} - \psi_j)^{-1} (\psi_j \Delta_j - \psi_{j-1}\Delta_{j-1})(\psi_j \varphi_j - \psi_{j-1}\varphi_{j-1})) - R_j \Delta_j \psi_j^{\frac{1}{\theta}}],$$

$$\frac{\partial^2 lnL}{\partial P \partial \theta} = C \sum_{j=1}^{k} (lnS_j)[r_j(\psi_{j-1} - \psi_j)^{-1}(\psi_j \Delta_j ((\varphi_j - \tau_j) + \frac{1}{\theta}(1 - \psi_j^{\frac{1}{\theta}})) - \psi_{j-1}\Delta_{j-1}((\varphi_{j-1} - \tau_{j-1}) + \frac{1}{\theta}(1 - \psi_{j-1}^{\frac{1}{\theta}})) - (\psi_{j-1} - \psi_j)^{-1} (\psi_j \Delta_j - \psi_{j-1}\Delta_{j-1})((\psi_{j-1}(\varphi_{j-1} - \tau_{j-1}) - \psi_j(\varphi_j - \tau_j))) - \frac{1}{\theta} R_j \Delta_j (1 - \psi_j^{\frac{1}{\theta}})],$$

$$\frac{\partial^2 lnL}{\partial \gamma \partial \theta} = \frac{\theta}{\gamma} \sum_{j=1}^{k} [r_j(\psi_{j-1} - \psi_j)^{-1}(\psi_j \varphi_j (\varphi_j - \tau_j - \frac{1}{\theta}\psi_j^{\frac{1}{\theta}}) - \psi_{j-1}\varphi_{j-1}(\varphi_{j-1} - \tau_{j-1} - \frac{1}{\theta}\psi_{j-1}^{\frac{1}{\theta}}) - (\psi_{j-1} - \psi_j)^{-1} (\psi_j \varphi_j - \psi_{j-1}\varphi_{j-1})((\psi_{j-1}(\varphi_{j-1} - \tau_{j-1}) - \psi_j(\varphi_j - \tau_j))) - \frac{1}{\theta} R_j \varphi_j (1 - \psi_j^{\frac{1}{\theta}})].$$

Therefore, the elements of Fisher information matrix for the MLE can be obtained as follows, i.e.,

$$F = \begin{pmatrix} f_{11} & f_{12} & f_{13} & f_{14} \\ & f_{22} & f_{23} & f_{24} \\ & & f_{33} & f_{34} \\ & & & f_{44} \end{pmatrix} = -E \begin{pmatrix} \frac{\partial^2 lnL}{\partial c^2} & \frac{\partial^2 lnL}{\partial c \partial p} & \frac{\partial^2 lnL}{\partial c \partial \gamma} & \frac{\partial^2 lnL}{\partial c \partial \theta} \\ & \frac{\partial^2 lnL}{\partial p^2} & \frac{\partial^2 lnL}{\partial p \partial \gamma} & \frac{\partial^2 lnL}{\partial p \partial \theta} \\ & & \frac{\partial^2 lnL}{\partial \gamma^2} & \frac{\partial^2 lnL}{\partial \gamma \partial \theta} \\ & & & \frac{\partial^2 lnL}{\partial \theta^2} \end{pmatrix}.$$

The asymptotic variance of the estimated model parameters $\hat{C}$,$\hat{P}$, $\hat{\gamma}$, and $\hat{\theta}$ can be obtained from the asymptotic variance-covariance matrix, which is defined as the inverse matrix of Fisher information matrix, i.e, $\Sigma = \hat{F}^{-1}$ where,

$$\hat{F} = -\begin{pmatrix} \frac{\partial^2 lnL}{\partial c^2} & \frac{\partial^2 lnL}{\partial c \partial p} & \frac{\partial^2 lnL}{\partial c \partial \gamma} & \frac{\partial^2 lnL}{\partial c \partial \theta} \\ & \frac{\partial^2 lnL}{\partial p^2} & \frac{\partial^2 lnL}{\partial p \partial \gamma} & \frac{\partial^2 lnL}{\partial p \partial \theta} \\ & & \frac{\partial^2 lnL}{\partial \gamma^2} & \frac{\partial^2 lnL}{\partial \gamma \partial \theta} \\ & & & \frac{\partial^2 lnL}{\partial \theta^2} \end{pmatrix} \downarrow (\hat{C}, \hat{P}, \hat{\gamma}, \hat{\theta}).$$

As the sample size N becomes large, the MLE under certain regularity conditions are consistent and asymptotically normally distributed. Thus a $(1-\alpha)100$ confidence interval of the parameter $\hat{C}$,$\hat{P}$, $\hat{\gamma}$, and $\hat{\theta}$ can be obtained such that

$$\hat{\Phi} \pm z_{\alpha \backslash 2} \sqrt{AVar(\hat{\Phi})}, \text{ where } \hat{\Phi} \text{ may be } \hat{C}, \hat{P}, \hat{\gamma}, \text{ or } \hat{\theta}.$$

To predict the value of the scale parameter α_u under the usual condition stress V_u, the invariance property of MLE is used. Thus, the MLE of $\hat{\alpha_u}$ and $\hat{R}_u(x_0)$ at the lifetime x_0 under the usual condition stress V_u, respectively, are given by the following equations $\hat{\alpha} = \hat{C} S_u^{\hat{P}}$, where $S_u = \frac{V^*}{V_u}, V^* = \prod_{j=1}^{k} V_j^{b_j}$, $b_j = \frac{n_j}{\sum_{j=1}^{k} n_j}$ and $\hat{R}_u(x_0) = (1 + \frac{\gamma}{\theta} e^{\hat{\alpha}_u x_0})^{-\theta}$.

4.Optimum Test Plan

In this Section, we determine the optimal value of x, the length of the inspection intervals in a k-level CSALT with progressive type-I group censoring according to the A-Optimality criterion. This criterion is based on the maximizing the sum of the diagonal entries of Fisher information matrix. Therefore, the optimal value of the length of the inspection intervals can be obtained by solving the following equation $\frac{\partial f}{\partial x} = 0$, where $f = f_{11} + f_{22} + f_{33} + f_{44}$, the sum of the diagonal elements of the Fisher information matrix. Therefore,

$\frac{\partial f}{\partial x} = f'_{11} + f'_{22} + f'_{33} + f'_{44}$, where
$f'_{11} = \sum_{j=1}^{k} S_j^P[r_j(\psi_{j-1}-\psi_j)^{-1}[[(j\psi_j\Delta_j(\frac{C}{\gamma}\Delta_j e^{-jCS_j^P x}+(1-\frac{1}{\gamma}\Delta_j e^{-jCS_j^P x})(C\Delta_j(2\psi_j^{\frac{1}{\theta}}-$
$\theta\varphi_j)+2\theta\varphi_j)))-((j-1)\psi_{j-1}\Delta_{j-1}(\frac{C}{\gamma}\Delta_{j-1}e^{-(j-1)CS_j^P x}+(1-(\frac{1}{\gamma}\Delta_{j-1}e^{-(j-1)CS_j^P x})(C\Delta_{j-1}(2\psi_{j-1}^{\frac{1}{\theta}}-$
$\theta\varphi_{j-1}) + 2\theta\varphi_{j-1})))) + (\psi_{j-1} - \psi_j)^{-1}(\psi_j\Delta_j - \psi_{j-1}\Delta_{j-1})[2[j\psi_j(C\Delta_j(\psi_j^{\frac{1}{\theta}} - \theta\varphi_j) +$
$\theta\varphi_j) - (j-1)\psi_{j-1}(C\Delta_{j-1}(\psi_{j-1}^{\frac{1}{\theta}} - \theta\varphi_{j-1}) + \theta\varphi_{j-1})] - C\theta(\psi_{j-1} - \psi_j)^{-1}(j\psi_j\varphi_j - (j -$
$1)\psi_{j-1}\varphi_{j-1})(\psi_j\Delta_j - \psi_{j-1}\Delta_{j-1})]] - C\theta(\psi_{j-1}-\psi_j)^{-1}(j\psi_j\varphi_j-(j-1)\psi_{j-1}\varphi_{j-1})[\psi_j\Delta_j^2(1-$
$\frac{1}{\gamma}e^{-jCS_j^P x}) - \psi_{j-1}\Delta_{j-1}^2(1 - \frac{1}{\gamma}e^{-(j-1)CS_j^P x}) + (\psi_{j-1} - \psi_j)^{-1}(\psi_j\Delta_j - \psi_{j-1}\Delta_{j-1})^2]] +$
$\frac{1}{\gamma}jR_j\Delta_j e^{-jCS_j^P x}[2\theta\varphi_j - C\Delta_j(1 - 2\psi_j^{\frac{1}{\theta}})]]$,
$f'_{22} = C\sum_{j=1}^{k} S_j^P(lnS_j)^2[r_j(\psi_{j-1}-\psi_j)^{-1}[j\psi_j[C\Delta_j(\frac{C}{\gamma}\Delta_j e^{-jCS_j^P x}+(1-\frac{1}{\gamma}e^{-jCS_j^P x})(\theta\varphi_j+$
$C\Delta_j\psi_j^{\frac{1}{\theta}}))+(C\Delta_j(1-\frac{1}{\gamma}e^{-jCS_j^P x})-1)(C\Delta_j(\psi_j^{\frac{1}{\theta}}-\theta\varphi_j)+\theta\varphi_j)]-(j-1)\psi_{j-1}[C\Delta_{j-1}(\frac{C}{\gamma}\Delta_j e^{-(j-1)CS_j^P x}+$
$(1-\frac{1}{\gamma}e^{-(j-1)CS_j^P x})(\theta\varphi_{j-1}+C\Delta_{j-1}\psi_{j-1}^{\frac{1}{\theta}}))+(C\Delta_{j-1}(1-\frac{1}{\gamma}e^{-(j-1)CS_j^P x})-1)(C\Delta_{j-1}(\psi_{j-1}^{\frac{1}{\theta}}-$
$\theta\varphi_{j-1}) + \theta\varphi_{j-1})] + C(\psi_{j-1} - \psi_j)^{-1}(\psi_j\Delta_j - \psi_{j-1}\Delta_{j-1})(2[j\psi_j(C\Delta_j(\psi_j^{\frac{1}{\theta}} - \theta\varphi_j) +$
$\theta\varphi_j) - (j-1)\psi_{j-1}(C\Delta_{j-1}(\psi_{j-1}^{\frac{1}{\theta}} - \theta\varphi_{j-1}) + \theta\varphi_{j-1})] - C\theta(\psi_{j-1} - \psi_j)^{-1}(j\psi_j\varphi_j - (j -$
$1)\psi_{j-1}\varphi_{j-1})(\psi_j\Delta_j-\psi_{j-1}\Delta_{j-1}))-C\theta(\psi_{j-1}-\psi_j)^{-1}(j\psi_j\varphi_j-(j-1)\psi_{j-1}\varphi_{j-1})[\psi_j\Delta_j(C\Delta_j(1-$
$\frac{1}{\gamma}e^{-jCS_j^P x})-1) - \psi_{j-1}\Delta_{j-1}(C\Delta_{j-1}(1-\frac{1}{\gamma}e^{-(j-1)CS_j^P x})-1)+C(\psi_{j-1}-\psi_j)^{-1}(\psi_j\Delta_j -$
$\psi_{j-1}\Delta_{j-1})^2]] + jR_j(\frac{C}{\theta}\Delta_j e^{-jCS_j^P x}(C\Delta_j(\psi_j^{\frac{1}{\theta}} - 1) + \theta\varphi_j) + (1 + \frac{C}{\gamma}\Delta_j e^{-jCS_j^P x})(\theta\varphi_j +$
$C\psi_j^{\frac{1}{\theta}}\Delta_j))]$, $f'_{33} = \frac{C\theta}{\gamma^2}\sum_{j=1}^{k} S_j^P[r_j(\psi_{j-1}-\psi_j)^{-1}[j\psi_j\varphi_j[\psi_j^{\frac{1}{\theta}}\varphi_j(\theta+1)-(\theta\varphi_j-\psi_j^{\frac{1}{\theta}})^2]-(j-$
$1)\psi_{j-1}\varphi_{j-1}[\psi_{j-1}^{\frac{1}{\theta}}\varphi_{j-1}(\theta+1)-(\theta\varphi_{j-1}-\psi_{j-1}^{\frac{1}{\theta}})^2]+\theta(\psi_{j-1}-\psi_j)^{-1}(\psi_j\varphi_j-\psi_{j-1}\varphi_{j-1})(j\psi_j\varphi_j(\psi_j^{\frac{1}{\theta}}-$
$\theta\varphi_j) - (j-1)\psi_{j-1}\varphi_{j-1}(\psi_{j-1}^{\frac{1}{\theta}} - \theta\varphi_{j-1}) - \theta(\psi_{j-1} - \psi_j)^{-1}(\psi_j\varphi_j - \psi_{j-1}\varphi_{j-1})(j\psi_j\varphi_j -$
$(j-1)\psi_{j-1}\varphi_{j-1})) + (1 + \theta(\psi_{j-1} - \psi_j)^{-1}(\psi_j\varphi_j - \psi_{j-1}\varphi_{j-1}))[j\psi_j\varphi_j(\psi_j^{\frac{1}{\theta}} - \theta\varphi_j) - (j -$
$1)\psi_{j-1}\varphi_{j-1}(\psi_{j-1}^{\frac{1}{\theta}} - \theta\varphi_{j-1})] - \theta(\psi_{j-1} - \psi_j)^{-1}(j\psi_j\varphi_j - (j-1)\psi_{j-1}\varphi_{j-1})[\psi_j\varphi_j$
$(\theta\varphi_j - \psi_j^{\frac{1}{\theta}}) - \psi_{j-1}\varphi_{j-1}(\theta\varphi_{j-1} - \psi_{j-1}^{\frac{1}{\theta}}) + (\psi_j\varphi_j - \psi_{j-1}\varphi_{j-1})(1+\theta(\psi_{j-1}-\psi_j)^{-1}(\psi_j\varphi_j -$
$\psi_{j-1}\varphi_{j-1}))]] - jR_j\psi_j^{\frac{1}{\theta}}\varphi_j((1 - \psi_j^{\frac{1}{\theta}}) + \varphi_j)]$,
$f'_{44} = \frac{C}{\theta}\sum_{j=1}^{k} S_j^P[r_j(\psi_{j-1}-\psi_j)^{-1}[j\psi_j\varphi_j(\psi_j^{\frac{1}{\theta}}\varphi_j+(1-\psi_j^{\frac{1}{\theta}})(\psi_j^{\frac{1}{\theta}}-\theta\varphi_j))-(j-1)\psi_{j-1}\varphi_{j-1}(\psi_{j-1}^{\frac{1}{\theta}}\varphi_{j-1}+$
$(1-\psi_{j-1}^{\frac{1}{\theta}})(\psi_{j-1}^{\frac{1}{\theta}}-\theta\varphi_{j-1}))+\theta[j\psi_j\varphi_j(\varphi_j-\tau_j)(2(\psi_j^{\frac{1}{\theta}}-1)-\theta(\varphi_j-\tau_j))-(j-1)\psi_{j-1}\varphi_{j-1}(\varphi_{j-1}-$
$\tau_{j-1})(2(\psi_{j-1}^{\frac{1}{\theta}} - 1) - \theta(\varphi_{j-1} - \tau_{j-1}))] + \theta(\psi_{j-1} - \psi_j)^{-1}(\psi_j(\varphi_j - \tau_j) - \psi_{j-1}(\varphi_{j-1} -$
$\tau_{j-1}))[2(j\psi_j\varphi_j(\theta(\varphi_j - \tau_j) + (\psi_j^{\frac{1}{\theta}} - 1)) - (j-1)\psi_{j-1}\varphi_{j-1}(\theta(\varphi_{j-1} - \tau_{j-1}) + (\psi_{j-1}^{\frac{1}{\theta}} -$
$1))) - \theta(\psi_{j-1} - \psi_j)^{-1}(\psi_j(\varphi_j - \tau_j) - \psi_{j-1}(\varphi_{j-1} - \tau_{j-1}))(j\psi_j\varphi_j - (j-1)\psi_{j-1}\varphi_{j-1})] -$
$\theta(\psi_{j-1} - \psi_j)^{-1}(j\psi_j\varphi_j - t(j-1)\psi_{j-1}\varphi_{j-1})[\psi_j\varphi_j(1 - \psi_j^{\frac{1}{\theta}}) - \psi_{j-1}\varphi_{j-1}(1 - \psi_{j-1}^{\frac{1}{\theta}}) +$
$\theta(\psi_j(\varphi_j - \tau_j)^2 - \psi_{j-1}(\varphi_{j-1} - \tau_{j-1})^2) + \theta(\psi_{j-1} - \psi_j)^{-1}$
$(\psi_j(\varphi_j - \tau_j) - \psi_{j-1}(\varphi_{j-1} - \tau_{j-1}))^2]] - jR_j\varphi_j\psi_j^{\frac{1}{\theta}}(\varphi_j + (1 - \psi_j^{\frac{1}{\theta}}))$.

The numerical solution of equation ($\frac{\partial f}{\partial x} = 0$) is obtained in order to get the optimal value of the length of the inspection intervals, x as will be shown in Section (5).

5.Numerical studies

This Section presents the numerical solutions to obtain the MLE of the unknown parameters C, P, γ, and θ, their mean squared errors (MSE), relative absolute biases (RAB), the Lower Bound (LB), the Upper Bound (UB), the estimated scale parameter α, and reliability function $R(x_0)$ under normal use conditions V_u. The RAB (the absolute difference between the estimated parameter and its true value divided by its true value), and the MSE (the mean square of the difference between the estimated parameter and its true value) are defined by the following equations, respectively. $RAB = \mid \frac{\hat{\Phi}-\Phi_0}{\Phi_0} \mid, MSE = \frac{\sum(\hat{\Phi}-\Phi_0)^2}{N}$, where Φ may be C, P, γ or θ. Also, this Section presents the numerical solutions to determine the best choice values of the length of the inspection intervals corresponding to progressive type-I grouped censoring data. We considered a progressively type-I group censored sample under constant stress test to estimate the MLE, and the scale parameter and the reliability function at the design stress $V_u = 0.5$. Tables (1, 2) summarize the results of the simulation study based on $n_1 = 30$, $n_2 = 30$, $n_3 = 30$, $r_1 = 7$, $r_2 = 5$, $r_3 = 2$, $V_1 = 0.75$, $V_2 = 1.5$, and $V_3 = 2$. The results show that the MSE of the scale parameter $\alpha_j, j = 1, 2, 3$ decreases as the stress value $V_j, j = 1, 2, 3$. In addition, we note the variance of C and the variance of θ are the smallest and converge to zero. On the other hand, the reliability decreases when the mission time x_0 increases. Moreover, there is an inverse proportional relationship between $\hat{\alpha}_u$ and $\hat{R}_u(x_0)$ at the same mission time.

Table 1: The MLE, RAB, and MSE

Case 1: C_0=0.5, P_0=1.0, γ_0=1.0, θ_0=1.0,α_{10}=0.7368,α_{20}=0.3684, α_{30}=0.2763							
Para.	C	P	γ	θ	α_1	α_2	α_3
MLE	0.5626	0.9749	1.0529	1.0570	0.8209	0.4177	0.3155
RAB	0.1251	0.0251	0.0529	0.0570	0.1142	0.1338	0.1420
MSE	0.0039	0.0006	0.0028	0.0032	0.0071	0.0024	0.0015
Case 2: C_0=0.65, P_0=0.7, γ_0=1.0, θ_0=0.9,α_{10}=0.8526,α_{20}=0.5249, α_{30}=0.4291							
MLE	0.6044	0.7101	0.9708	0.8647	0.7960	0.4866	0.3967
RAB	0.0701	0.0144	0.02920	0.0392	0.0665	0.0730	0.0757
MSE	0.0021	0.0001	0.0009	0.0012	0.0032	0.0015	0.0011
Case 3: C_0=0.7, P_0=0.7,, γ_0=0.7,, θ_0=0.7,α_{10}=0.9182,α_{20}=0.5652, α_{30}=0.4621							
MLE	0.7467	0.6912	0.7191	0.7223	0.9761	0.6046	0.4955
RAB	0.0667	0.0126	0.0273	0.0319	0.0630	0.0696	0.0723
MSE	0.0022	0.0001	0.0004	0.0005	0.0034	0.0015	0.0011
Case 4: C_0=0.7, P_0=0.8, γ_0=0.8, θ_0=0.8,α_{10}=0.9545,α_{20}=0.5482, α_{30}=0.5482							
MLE	0.6991	0.8002	0.7996	0.7995	0.9534	0.5475	0.4349
RAB	0.0013	0.0003	0.0005	0.0006	0.0012	0.0013	0.0014
MSE	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
Case 5: C_0=0.7, P_0=0.8, γ_0=0.7, θ_0=0.8 ,α_{10}=0.9545,α_{20}=0.5482, α_{30}=0.4355							
MLE	0.7275	0.7935	0.7123	0.8143	0.9896	0.5709	0.4544
RAB	0.0393	0.0081	0.0175	0.0179	0.0367	0.0414	0.0434
MSE	0.0008	0.0000	0.0002	0.0002	0.0012	0.0005	0.0004

Table 2: Asymptotic Var-Cov Matrix, the Confidence Intervals, and estimates of α and $R(x_0)$ under V_u

Case 1: C_0=0.5, P_0=1.0, γ_0=1.0, θ_0=1.0								
Para.	Var-Cov Matrix				CI	$\hat{\alpha}_u$	x_0	$\hat{R}_u(x_0)$
C	0.0024	0.0017	0.0245	0.0391	0.4200-0.5800	1.2189	0.005	0.4801
P		0.0617	-0.0548	-0.0905	0.5914-1.4086		0.5	0.3327
γ			0.0265	0.0918	0.7323-1.2677		1.5	0.2104
θ				0.0851	0.5201-1.4799		2.0	0.0698
Case 2: C_0=0.65, P_0=0.7, γ_0=1.0, θ_0=0.9								
C	0.0006	0.0078	0.0186	0.0307	0.6086-0.6914	1.0615	0.005	0.5203
P		0.0450	-0.0357	-0.0622	0.3509-1.0491		0.5	0.3972
γ			0.0039	0.0596	0.8969-1.1031		1.5	0.2865
θ				0.0533	0.5202-1.2798		2.0	0.1322
Case 1: C_0=0.5, P_0=1.0, γ_0=1.0, θ_0=1.0								
Para.	Var-Cov Matrix				CI	$\hat{\alpha}_u$	x_0	$\hat{R}_u(x_0)$
Case 3: C_0=0.7, P_0=0.7, γ_0=0.7, θ_0=0.7								
C	0.0034	0.0067	0.0224	0.0381	0.6036-0.7964	1.2918	0.005	0.6057
P		0.0442	-0.0339	-0.0583	0.3540-1.0460		0.5	0.4635
γ			0.0041	0.0521	0.5952-0.8048		1.5	0.3309
θ				0.0540	0.3177-1.0823		2.0	0.1472
Case 4: C_0=0.7, P_0=0.8, γ_0=0.8, θ_0=0.8								
C	0.0036	0.0043	0.0242	0.0400	0.6008-0.7992	1.3188	0.005	0.5730
P		0.505	-0.04044	-0.0682	0.4303-1.1697		0.5	0.4229
γ			0.0102	0.0630	0.6341-0.9659		1.5	0.2882
θ				0.0641	0.3836-1.2164		2.0	0.1149
Case 5: C_0=0.7, P_0=0.8, γ_0=0.7, θ_0=0.8								
C	0.0045	0.0040	0.0269	0.0437	0.5894-0.8106	1.3652	0.005	0.5979
P		0.0486	-0.0426	-0.0695	0.4375-1.1625		0.5	0.4413
γ			0.0169	0.0681	0.4860-0.9140		1.5	0.2978
θ				0.0704	0.3636-1.2364		2.0	0.1138

The optimal value of the inspection intervals x^* is obtained according to the A-optimality criteria. Table (3) shows different values of x^* and GAV for $k = 1, 2$ at different initial guesses values of C, P, γ, θ and different sample sizes. The stress levels to the test units are $V_1 = 1$, $V_2 = 2$. From the results, we observe GAV decreases as the sample size increases.

Table 3: Optimum Values of x^* and GAV.

Case 1: $C_0 = 1.0$, $P_0 = 1.0$, $\gamma_0 = 1.0$, θ_0=1.0							Case 2: $C_0 = 1.0$, $P_0 = 1.2$, $\gamma_0 = 1.0$, θ_0=1.2						
N	n_1	n_2	r_1	r_2	$x*$	GAV	N	n_1	n_2	r_1	r_2	$x*$	GAV
45	30	15	3	2	0.930	0.00002222	45	30	15	3	2	0.747	0.00002056
100	75	25	10	5	0.946	0.00000109	100	75	25	10	5	0.903	0.00000108
200	125	75	15	10	0.919	0.00000005	200	125	75	15	10	0.697	0.00000005
300	200	100	22	10	1.003	0.00000002	300	200	100	22	10	0.953	0.00000000
400	200	200	30	15	0.986	0.00000000	400	200	200	30	15	0.675	0.00000000
500	300	200	27	13	1.021	0.00000000	500	300	200	27	13	0.822	0.00000000
Case 3: $C_0 = 0.8$, $P_0 = 1.0$, $\gamma_0 = 1.3$, θ_0=1.2							Case 4: $C_0 = 1.0$, $P_0 = 1.3$, $\gamma_0 = 1.3$, θ_0=1.3						
45	30	15	3	2	0.297	0.00002181	45	30	15	3	2	0.348	0.00001396
100	75	25	10	5	0.318	0.00000111	100	75	25	10	5	0.385	0.00000073
200	125	75	15	10	0.282	0.00000001	200	125	75	15	10	0.333	0.00000003
300	200	100	22	10	0.336	0.00000000	300	200	100	22	10	0.407	0.00000001
400	200	200	30	15	0.290	0.00000000	400	200	200	30	15	0.347	0.00000000
500	300	200	27	13	0.333	0.00000000	500	300	200	27	13	0.399	0.00000000

6.Conclusion

This paper dealt with constant stress accelerated life testing in the case of progressive type I censoring using grouped data. It is assumed that the lifetime of test units follow the generalized logistic distribution. The generalized logistic distribution is more appropriate distribution to model the failure time in a constant stress accelerated life testing. Maximum likelihood estimators of the model parameters and the associated Fisher Information matrix are derived. For various combinations of the model parameters, the corresponding optimum length of the inspection interval are obtained numerically using the A-optimality criteria. Some interesting findings are discussed. The most important result is that the length of the inspection interval is increased when GAV decreases and the sample size increases. In the future, this work may be extended to investigate "progressive step-stress" testing in which stress is increased step-by-step over time. It may also be extended to deal with random removals, such that the number of removed units in each stage are not pre-fixed.

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