FEKETE SZEGÖ PROBLEM FOR CLASSES OF ANALYTIC FUNCTIONS DEFINED BY Q- OPERATOR

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ABSTRACT

Motivating by q analogue theory, q analogue of a linear operator has been defined and subclass of analytic functions is introduced in this work. The Fekete-Szegö problem for this class is obtained. Various Known or special cases of our results are pointed out.

***Keywords*:** q-analogue, analytic function, Hohlov operator,Fekete-Szegö inequality.

1. INTRODUCTION

The theory of q-analogues or q-extensions of classical formulas and functions based on the observation that

$$\lim_{q\to 1}\frac{1-q^{\alpha}}{1-q}=\alpha, |q|<1,$$

therefore the number $(1-q^{\alpha})/(1-q)$ is sometimes called the basic number $[\alpha]_q$.

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z)=z+\sum_{k=2}^{\infty} a_k z^k a_k \geq 0, (1)$$

which are analytic in the open unit disk $\mathbb{U}=\{z\in\mathbb{C}:|z|<1\}$.

Let $f,g\in\mathcal{A}$, where f is defined by (1.1) and g is given by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k, \quad b_k \geq 0.$$

Then the Hadamard product (or convolution) $f * g$ of the functions f and g is defined by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k.$$

For complex parameters $a, b, c, q (c \in \mathbb{C} \backslash \{0, -1, -2, \dots\}, |q| < 1)$, we define the q- analogue of Gauss's hypergeometric function ${}_2\Phi_1(a, b; c, q, z)$ by

$${}_2\Phi_1(a, b; c, q, z) = \sum_{k=0}^{\infty} \frac{(a,q)_k (b,q)_k}{(q,q)_k (c,q)_k} z^k, \quad (z \in \mathbb{U})$$

(2)

where $(x, q)_k$ is the q-analogue of Pochhammer symbol defined by

$$(x, q)_k = \begin{cases} 1, & k = 0; \\ (1-x)(1-xq)(1-xq^2)\dots(1-xq^{k-1}), & k \in \mathbb{N}. \end{cases}$$

Note that $\lim_{q \to 1} [{}_2\Phi_1(a, b; c, q, z)] = F_1(a, b; c, z)$, where ${}_2F_1(a, b; c, z)$ is the familiar Gauss's hypergeometric function.

By using the ratio test, one recognize that, if $|q| < 1$, the q- analogue of Gauss's hypergeometric series(also called basic hypergeometric series or q-hypergeometric series) defined in (1.2) converges absolutely for $|z| < 1$.

For more mathematical background concern the q-analogue theory one may refer to [1, 2].

Corresponding to the function ${}_2\Phi_1(a, b; c, q, z)$ given by (1.2), consider

$$\mathcal{G}_1(a, b; c, q, z) = z\,\Phi_1(a, b; c, q, z).$$

Now, we define the q-analogue operator $\mathcal{D}_q(a, b, c)f: \mathcal{A} \to \mathcal{A}$ of Hohlov linear operator as follows:

$$\begin{aligned} \mathcal{D}_q(a, b, c)f(z) &=_2 \mathcal{G}_1(a, b; c, q, z) * f(z) \\ &= z + \sum_{k=2}^{\infty} \nabla_q^k(a, b, c) a_k z^k, \end{aligned} \qquad (3)$$

where

$$\nabla_q^k(a, b, c) = \frac{(a,q)_{(k-1)}(b,q)_{k-1}}{(q,q)_{k-1}(c,q)_{k-1}} \quad (4)$$

For $0 \le \alpha \le 1$ and $0 \le \beta \le 1$. The class $W_q^{\alpha,\beta}(a, b, c, \varphi)$ consists of all analytic functions $f \in \mathcal{A}$ satisfying:

$$\left(\frac{z(\mathcal{D}_q(a,c,d)f(z))'}{\mathcal{D}_q(a,c,d)f(z)}\right)^{\alpha} \left(1 + \frac{z(\mathcal{D}_q(a,c,d)f(z))''}{(\mathcal{D}_q(a,c,d)f(z))'}\right)^{\beta} \prec \varphi(z)\,. \qquad (5)$$

We note that for suitable choices of $\alpha, \beta, a, b, c, q$ and $\varphi(z)$, we obtain the following subclasses:

1. $W_q^{1,0}(q, b, b, \varphi(z)) = S^*(\varphi)$ (see Ma and Minda [6]),
2. $W_q^{0,1}(q, b, b, \varphi(z)) = C^*(\varphi)$ (see Ma and Minda [6]),
3. $W_q^{\alpha,\beta}(q, b, b, \varphi(z)) = M_{\alpha,\beta}(\varphi)$ (see Ravichandran [7]).

Also, we note that for $0 \leq \alpha \leq 1$, $0 \leq \beta \leq 1$ and $z \in \mathbb{U}$ we have the following:

1. Putting $a = q^a, b = q^b, c = q^c$ and $q \to 1$, we obtain

$$W_q^{\alpha,\beta}(q^a, q^b, q^c, \varphi(z)) = W_{\alpha,\beta}(a, b, c, \varphi) = \{f \in$$

$$\mathcal{A}: \left(\frac{z(H_{a,b,c}f(z))'}{h_{a,b,c}f(z)}\right)^{\alpha}$$

$$\left(1 + \frac{z(H_{a,b,c}f(z))''}{(H_{a,b,c}f(z))'}\right)^{\beta} \prec \varphi(z), \},$$

where the operator $H_{a,b,c}f(z)$ was defined by Hohlov [9].

2. Putting $a = q^{\delta+1}(\delta > -1), b = c = q$ and $q \to 1$, we obtain

$$W_q^{\alpha,\beta}(q^{\delta+1}, q, q, \varphi(z)) = W_{\alpha,\beta}^{\delta}(\varphi) = \{f \in \mathcal{A}: \left(\frac{z(D^{\delta}f(z))'}{D^{\delta}f(z)}\right)^{\alpha}$$

$$\left(1 + \frac{z(D^{\delta}f(z))''}{(D^{\delta}f(z))'}\right)^{\beta} \prec \varphi(z), \delta > -1\},$$

where the operator $D^{\delta}f(z)$ was defined by Ruscheweyh [3].

3. Putting $a = q^a, b = q, c = q^c, q \to 1$ and $a > 0, c > 0$ we obtain

$$W_q^{\alpha,\beta}(q^a, q, q^c, \varphi(z)) = W_{\alpha,\beta}(a, c, \varphi) = \{f \in \mathcal{A}: \left(\frac{z(L(a,c)f(z))'}{L(a,c)f(z)}\right)^{\alpha}$$

$$\left(1 + \frac{z(L(a,c)f(z))''}{(L(a,c)f(z))'}\right)^{\beta} \prec \varphi(z), a > 0, c > 0\},$$

where the operator $L(a,c)f(z)$ was defined by Carlson and Shaffer [4].

4. Putting $a = q^a, b = q^{\lambda+1}, c = q^c (\lambda > -1), q \to 1$ and $a, c \in \mathbb{R}\backslash\mathbb{Z}_0^-, \mathbb{Z}_0^- = \{0, -1, -2, \dots\}$, we obtain

$$W_q^{\alpha,\beta}(q^a, q^{\lambda+1}, q^c, \varphi(z)) = W_{\alpha,\beta}^{\lambda}(a, c, \varphi) = \{f \in \mathcal{A}: \left(\frac{z(I^\lambda f(z))'}{I^\lambda f(z)}\right)^\alpha$$

$$\left(1 + \frac{z(I^\lambda f(z))''}{(I^\lambda f(z))'}\right)^\beta \prec \varphi(z), a, c \in \mathbb{R}\backslash\mathbb{Z}_0^-, \lambda > -1\},$$

where the operator $I^\lambda f(z)$ was defined by Cho et al. [[11]with $p = 1$].

5. Putting $a = q^2, b = q, q \to 1$ and $c = q^{n+1} (n > -1)$ we obtain

$$W_q^{\alpha,\beta}(q^2, q, q^{n+1}, \varphi(z)) = W_{\alpha,\beta}(n, \varphi) = \{f \in \mathcal{A}: \left(\frac{z(I_n f(z))'}{I_n f(z)}\right)^\alpha$$

$$\left(1 + \frac{z(I_n f(z))''}{(I_n f(z))'}\right)^\beta \prec \varphi(z), n > -1\},$$

where the operator $I_n f(z)$ was introduced by Noor [8. 10].

6. Putting $a = q^{c+1}, b = q, c = q^{c+2}, (c > -1)$ and $q \to 1$, we obtain

$$W_q^{\alpha,\beta}(q^{c+1}, q, q^{c+2}, \varphi(z)) = W_{\alpha,\beta}(c, \varphi) = \{f \in \mathcal{A}: \left(\frac{z(J_n f(z))'}{J_n f(z)}\right)^\alpha$$

$$\left(1 + \frac{z(J_n f(z))''}{(J_n f(z))'}\right)^\beta \prec \varphi(z), n > -1\},$$

where the operator $J_n f(z)$ is the integral operator was defined by Bernardi [5].

7. Putting $a = q^2, b = q, c = q^{2-\lambda}, (\infty < \lambda < 2)$ and $q \to 1$, we obtain

$$W_q^{\alpha,\beta}(q^2, q, q^{2-\lambda}, \varphi(z)) = W_{\alpha,\beta}(\lambda, \varphi) = \{f \in \mathcal{A}: \left(\frac{z(\Omega_z^\lambda f(z))'}{\Omega_z^\lambda f(z)}\right)^\alpha$$

$$\left(1+\frac{z(\Omega_z^\lambda f(z))''}{(\Omega_z^\lambda f(z))'}\right)^\beta \prec \varphi(z), \infty < \lambda < 2\},$$

where the operator $\Omega^\lambda: \mathcal{A} \to \mathcal{A}$ was defined by Owa and Srivastava [12] and Owa [13] as:

$$\begin{aligned}(\Omega^\lambda f)(z) &= \Gamma(2-\lambda) z^\lambda D_z^\lambda f(z) \\ &= z + \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(2-\lambda)}{\Gamma(k+1-\lambda)} a_k z^k \quad (\lambda \neq 2,3,\dots).\end{aligned}$$

We denote by $\mathcal{P}$ a class of analytic functions in $\mathbb{U}$ with

$$p(z) = 1 \quad \text{and} \quad \Re(p(z)) > 0.$$

In order to derive our main results, we have to recall the following lemmas.

Lemma1. [14] Let $p \in \mathcal{P}$ with $p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \dots$. Then for any complex number

$$|p_2 - \mu p_1^2| \leq 2max\{1, |2\mu - 1|\},$$

and the result is sharp for the functions given by

$$p(z) = \frac{1+z^2}{1-z^2}, \quad p(z) = \frac{1+z}{1-z}.$$

Lemma2. [14] Let $p \in \mathcal{P}$ with $p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \dots$, then

$$|p_2 - \nu p_1^2| \leq \begin{cases} -4\nu + 2, & if \nu \leq 0; \\ 2, & if\ 0 \leq \nu \leq 1; \\ 4\nu - 2, & if \nu \geq 1. \end{cases}$$

When $\nu < 0$, the equality holds if and only if $p(z) = \frac{1+z}{1-z}$ or one of its rotations.If $0 < \nu < 1$, then the equality holds if and only if $p(z) = \frac{1+z^2}{1-z^2}$ or one of its rotations.

If $\nu = 0$, then the equality holds if and only if

$$p(z) = \left(\frac{1}{2} + \frac{1}{2}\lambda\right)\frac{1+z}{1-z} + \left(\frac{1}{2} - \frac{1}{2}\lambda\right)\frac{1-z}{1+z}(0 \leq \lambda \leq 1),$$

or one of its rotations. If $\nu = 1$,the equality holds if and only if

$$\frac{1}{p(z)} = \left(\frac{1}{2} + \frac{1}{2}\lambda\right)\frac{1+z}{1-z} + \left(\frac{1}{2} - \frac{1}{2}\lambda\right)\frac{1-z}{1+z}(0 \leq \lambda \leq 1),$$

or one of its rotations. The above upper bound is sharp and it can be improved as follows when $0 < \nu < 1$:

$$|p_2 - \nu p_1^2| + \nu|p_1|^2 \leq 2(0 < \nu \leq \frac{1}{2}),$$

and

$$|p_2 - \nu p_1^2| + (1 - \nu)|p_1|^2 \leq 2(\frac{1}{2} < \nu \leq 1).$$

Unless otherwise mentioned, we assume throughout this paper that $\nabla_q^k(a, b, c)$ is given by (1.4).

2. FEKETE-SZEGÖ PROBLEM

The Fekete-Szegö problem was introduced first in [20]. This inequality is all about estimating the best possible constant $C(\mu)$, for real or complex number μ such that $|a_3 - \mu a_2^2| \leq C(\mu)$, for every function $f \in A$.

Theorem. Let $\varphi(z) = 1 + B_1 z + B_2 z^2 + \ldots, B_1 > 1$. If f given by (1.1) in the class $W_q^{\alpha,\beta}(a,b,c,\varphi)$ and μ is a complex number, then

$$|a_3 - \mu a_2^2| \leq \frac{B_1}{2(\alpha+3\beta)\nabla_q^3(a,b,c)}$$

$$\max\{1, |\frac{B_2}{B_1} - B_1 \frac{[(\alpha+2\beta)^2 - 3(\alpha+4\beta)][\nabla_q^2(a,b,c)]^2 + 4\mu(\alpha+3\beta)\nabla_q^3(a,b,c)}{2(\alpha+2\beta)^2[\nabla_q^2(a,b,c)]^2}|\}. \quad (5)$$

The result is sharp.

Proof. If $f \in W_q^{\alpha,\beta}(a,b,c,\varphi)$, then there is a Schwarz function $w(z)$ in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ in $\mathbb{U}$ such that

$$\left(\frac{z(\mathcal{D}_q(a,c,d)f(z))'}{\mathcal{D}_q(a,c,d)f(z)}\right)^{\alpha}\left(1 + \frac{z(\mathcal{D}_q(a,c,d)f(z))''}{(\mathcal{D}_q(a,c,d)f(z))'}\right)^{\beta} = \varphi(w(z)). \quad (7)$$

Define the function $g(z)$ by

$$p(z) = \frac{1+w(z)}{1-w(z)} = 1 + p_1 z + p_2 z^2 + \ldots. \quad (8)$$

Since $w(z)$ is a Schwarz function, we see that $\mathcal{R}(p(z)) > 0$ and $p(0) = 1$. Define

$$g(z) = \left(\frac{z(\mathcal{D}_q(a,c,d)f(z))'}{\mathcal{D}_q(a,c,d)f(z)}\right)^{\alpha}\left(1 + \frac{z(\mathcal{D}_q(a,c,d)f(z))''}{(\mathcal{D}_q(a,c,d)f(z))'}\right)^{\beta} = 1 + d_1 z + d_2 z^2 + \ldots. \quad (6)$$

In view of (2.2), (2.3), and (2.4), we have

$g(z) = \varphi\left(\frac{p(z)-1}{p(z)+1}\right).$ (7)

Since

$$\frac{p(z)-1}{p(z)+1} = \frac{1}{2}\left[p_1 z + \left(p_2 - \frac{p_1^2}{2}\right)z^2 + \left(p_3 + \frac{p_1^3}{4} - p_1 p_2\right)z^3 + \ldots.\right].$$

Therefore, we have

$$\varphi\left(\frac{p(z)-1}{p(z)+1}\right) = 1 + \frac{1}{2}B_1 p_1 z + \left[\frac{1}{2}B_1\left(p_2 - \frac{p_1^2}{2}\right) + \frac{1}{4}B_2 p_1^2\right]z^2 + \ldots, \quad (11)$$

from last equation we obtain

$$d_1 = \frac{1}{2}B_1 p_1,$$

and

$$d_2 = \frac{1}{2}B_1\left(p_2 - \frac{p_1^2}{2}\right) + \frac{1}{4}B_2 p_1^2.$$

A computation shows that

$$\frac{z(\mathcal{D}_q(a,c,d)f(z))'}{\mathcal{D}_q(a,c,d)f(z)} = 1 + \nabla_q^2(a,b,c)a_2 z + \{2\nabla_q^2(a,b,c)a_3 - [\nabla_q^2(a,b,c)]^2 a_2^2\}z^2 + \ldots,$$

so that

$$\left(\frac{z(\mathcal{D}_q(a,c,d)f(z))'}{\mathcal{D}_q(a,c,d)f(z)}\right)^{\alpha} = 1 + \alpha\nabla_q^2(a,b,c)a_2 z + \left\{2\alpha\nabla_q^3(a,b,c)a_3 + \frac{(\alpha^2 - 3\alpha)}{2}[\nabla_q^2(a,b,c)]^2 a_2^2\right\}z^2 + \ldots.$$

Similarly, we have

$$1 + \frac{z(\mathcal{D}_q(a,c,d)f(z))''}{(\mathcal{D}_q(a,c,d)f(z))'} = 1 + 2\nabla_q^2(a,b,c)a_2 z + \{6\nabla_q^3(a,b,c)a_3 - 4[\nabla_q^2(a,b,c)]^2 a_2^2\}z^2 + \ldots,$$

So that

$$\left(1+\frac{z(\mathcal{D}_q(a,c,d)f(z))''}{(\mathcal{D}_q(a,c,d)f(z))'}\right)^{\beta} = 1+2\beta\nabla_q^2(a,b,c)a_2 z+\{6\beta\nabla_q^3(a,b,c)a_3+2\beta(\beta-3)[\nabla_q^2(a,b,c)]^2a_2^2\}z^2+\ldots,$$

Thus we obtain

$$\left(\frac{z(\mathcal{D}_q(a,c,d)f(z))'}{\mathcal{D}_q(a,c,d)f(z)}\right)^{\alpha}\left(1+\frac{z(\mathcal{D}_q(a,c,d)f(z))''}{(\mathcal{D}_q(a,c,d)f(z))'}\right)^{\beta}$$

$$=1+(\alpha+2\beta)\nabla_q^2(a,b,c)a_2z+\left[2(\alpha+3\beta)\nabla_q^3(a,b,c)a_3+\frac{(\alpha+2\beta)^2-3(\alpha+4\beta)}{2\left[\nabla_q^2(a,b,c)\right]^2a_2^2}\right]z^2+\cdots$$

Then, we see that

$$d_1=(\alpha+2\beta)\nabla_q^2(a,b,c)a_2$$

and

$$d_2=2(\alpha+3\beta)\nabla_q^3(a,b,c)a_3+\frac{(\alpha+2\beta)^2-3(\alpha+4\beta)}{2}[\nabla_q^2(a,b,c)]^2a_2^2,$$

or equivalently we have

$$a_2=\frac{B_1p_1}{2(\alpha+2\beta)\nabla_q^2(a,b,c)},$$

and

$$a_3=\frac{B_1p_2}{4(\alpha+3\beta)\nabla_q^3(a,b,c)}-\frac{p_1^2}{8(\alpha+3\beta)\nabla_q^3(a,b,c)}[B_1-B_2+\frac{[(\alpha+2\beta)^2-3(\alpha+4\beta)]}{2(\alpha+2\beta)^2}B_1^2].$$

Therefore

$$a_3-\mu a_2^2=\frac{B_1}{4(\alpha+3\beta)\nabla_q^3(a,b,c)}\{p_2-\nu p_1^2\},$$

where

$$\nu=\frac{1}{2}[1-\frac{B_2}{B_1}+\frac{[(\alpha+2\beta)^2-3(\alpha+4\beta)][\nabla_q^2(a,b,c)]^2+4\mu(\alpha+3\beta)\nabla_q^3(a,b,c)}{2(\alpha+2\beta)^2[\nabla_q^2(a,b,c)]^2}B_1],$$

the result follows by an application of Lemma 1.1 The result is sharp for functions

$$\left(\frac{z(\mathcal{D}_q(a,c,d)f(z))'}{\mathcal{D}_q(a,c,d)f(z)}\right)^{\alpha}\left(1+\frac{z(\mathcal{D}_q(a,c,d)f(z))''}{(\mathcal{D}_q(a,c,d)f(z))'}\right)^{\beta}=\varphi(z^2),$$

and

$$\left(\frac{z(\mathcal{D}_q(a,c,d)f(z))'}{\mathcal{D}_q(a,c,d)f(z)}\right)^{\alpha}\left(1+\frac{z(\mathcal{D}_q(a,c,d)f(z))''}{(\mathcal{D}_q(a,c,d)f(z))'}\right)^{\beta}=\varphi(z).$$

This completes the proof of theorem 2.1.

Taking $a=q$ and $b=c$ in Theorem 2.1, we get the following corollary.

Corollary1. Let $\varphi(z)=1+B_1z+B_2z^2+\ldots,B_1>1$. If f given by (1.1) in the class $M_{\alpha,\beta}(\varphi)$ and μ is a complex number, then

$$|a_3-\mu a_2^2|\leq\frac{B_1}{2(\alpha+3\beta)}$$

$$\max\{1,|\frac{B_2}{B_1}-B_1\frac{[(\alpha+2\beta)^2-3(\alpha+4\beta)]+4\mu(\alpha+3\beta)}{2(\alpha+2\beta)^2}|\}. \qquad (8)$$

The result is sharp.

By using Lemma 1. we obtain the following theorem.

Theorem2. Let $\varphi(z) = 1 + B_1 z + B_2 z^2 + \dots, (B_i > 0, i \in \mathbb{N})$. If f given by (1.1) in the class $W_q^{\alpha,\beta}(a,b,c,\varphi)$ and μ is a complex number, then

$$|a_3 - \mu a_2^2| \leq$$

$$\begin{cases} \dfrac{1}{2(\alpha+3\beta)\nabla_q^3(a,b,c)}\left\{B_2 - \dfrac{\gamma}{2(\alpha+2\beta)^2[\nabla_q^2(a,b,c)]^2}B_1^2\right\}, & if \mu \leq \sigma_1; \\ \dfrac{1}{2(\alpha+3\beta)\nabla_q^3(a,b,c)}, & if \sigma_1 \leq \mu \leq \sigma_2; \\ \dfrac{1}{2(\alpha+3\beta)\nabla_q^3(a,b,c)}\left\{-B_2 + \dfrac{\gamma}{2(\alpha+2\beta)^2[\nabla_q^2(a,b,c)]^2}B_1^2\right\}, & \mu \geq \sigma_2. \end{cases}$$

where

$$\sigma_1 = \frac{2(\alpha+2\beta)^2[\nabla_q^2(a,b,c)]^2(B_2 - B_1) - [(\alpha+2\beta)^2 - 3(\alpha+4\beta)][\nabla_q^2(a,b,c)]^2 B_1^2}{4(\alpha+3\beta)\nabla_q^3(a,b,c)B_1^2},$$

$$\sigma_2 = \frac{2(\alpha+2\beta)^2[\nabla_q^2(a,b,c)]^2(B_2 + B_1) - [(\alpha+2\beta)^2 - 3(\alpha+4\beta)][\nabla_q^2(a,b,c)]^2 B_1^2}{4(\alpha+3\beta)\nabla_q^3(a,b,c)B_1^2}.$$

and

$$\gamma = [(\alpha+2\beta)^2 - 3(\alpha+4\beta)][\nabla_q^2(a,b,c)]^2 + 4\mu(\alpha+3\beta)\nabla_q^3(a,b,c). \tag{9}$$

The result is sharp.

Proof. First, let $\mu \leq \sigma_1$, then

$$\begin{aligned}|a_3-\mu a_2^2| &\le \frac{B_1}{2(\alpha+3\beta)\nabla_q^3(a,b,c)}\left\{\frac{B_2}{B_1}-\frac{\gamma}{2(\alpha+2\beta)^2[\nabla_q^2(a,b,c)]^2}B_1\right\}\\ &\le \frac{1}{2(\alpha+3\beta)\nabla_q^3(a,b,c)}\left\{B_2-\frac{\gamma}{2(\alpha+2\beta)^2[\nabla_q^2(a,b,c)]^2}B_1^2\right\}.\end{aligned}$$

Let, now $\sigma_1 \le \mu \le \sigma_2$. Then, using the above calculation, we obtain

$$|a_3-\mu a_2^2| \le \frac{B_1}{2(\alpha+3\beta)\nabla_q^3(a,b,c)}.$$

Finally, if $\mu \ge \sigma_2$, then

$$\begin{aligned}|a_3-\mu a_2^2| &\le \frac{B_1}{2(\alpha+3\beta)\nabla_q^3(a,b,c)}\left\{-\frac{B_2}{B_1}+\frac{\gamma}{2(\alpha+2\beta)^2[\nabla_q^2(a,b,c)]^2}B_1\right\}\\ &\le \frac{1}{2(\alpha+3\beta)\nabla_q^3(a,b,c)}\left\{-B_2+\frac{\gamma}{2(\alpha+2\beta)^2[\nabla_q^2(a,b,c)]^2}B_1^2\right\}.\end{aligned}$$

Following the technique in [[15], Theorem 2.1], it can be showing that the result is sharp.

Taking $a=q$ and $b=c$ in Theorem 2, we obtain the result obtained by Ravichandran [[7], Theorem 2.1].

By specializing the parameters α, β, a, b, c and q in Theorems 1 and 2, we can obtain new results corresponding to the operators mentioned in the introduction. Other work related to Fekete Szegö Problem and subordination can be found in[16, 17]. Also work related to basic hypergeometric functions can be found in [18. 19].

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