APPLICATION OF THE VARIATIONAL ITERATION METHOD AND PERTURBATION METHOD FOR THE FOAM DRAINAGE EQUATION

Mohamed M. Alargat , Najla J. Alawis*, Mariam R. Abusetta,

Mathematics Department, Science Faculty, Al-Asmarya University, Zliten, Libya
**Corresponding author: ngla111@yahoo.com*

ABSTRACT

We extend he's Variational Iteration Method and Perturbation method, applied to some systems of nonlinear (PDEs). In this paper, we consider Foam Drainage Equation (FDE) are a prime example of a multiphase "soft condensed matter". We find both exact and approximate solutions, we compared the numerical solutions with corresponding analytical solution.

Key words: He's Variational Iteration Method, Perturbation method Foam Drainage Equation, Numerical solution .

1. INTRODUCTION

Foam are of great importance in many technological processes and applications, and their properties are subject of intensive studies from both practical and seientific point of view [10]. liquid foam is an example of soft matter (or acomplex fluid) with avery well defined structure first clearly described by joseph plateau in the 19th century.

In this work we consider, he's Variational iteration method as a well known method for finding both analytic and approximate solutions of PDE, and Perturbation method. It is well known that there are many system of nonlinear partial equations in the study of physics machanies and biologics.

In this paper, we consider Foam Drainage Equation (FDE) is $A_t + (A^2 - \frac{\sqrt{A}}{2} A_x)_x = 0,$ where x and t are scaled position and time coordinates respectively. In the case of forced drainage, the solution can be expressed as :

$$A(x,t) = c\tanh^2\left(\sqrt{c}(x-ct)\right), \qquad ,x \le ct$$

$$= 0 \qquad ,x > ct,$$

where c is the velocity of the wave front .

many powerful methods have been presented, for instance by adomian decomposition method and the tanh method[8] for solving of (FDE).

2. APPLYING VARIATIONAL ITERATION METHOD (VIM)

We will study the following non-linear foam drainage equation:

$$A_t + (A^2 - \frac{\sqrt{A}}{2} A_x)_x = 0, \qquad (1)$$

where x and t are scaled position and time coordinates respectively.

In the case of forced drainage, the solution can be expressed as [2,8]:

$$A(x,t) = c\tanh^2\left(\sqrt{c}(x-ct)\right), \qquad ,x \le ct \qquad (2)$$

$$= 0 \qquad ,x > ct,$$

where c is the velocity of the wave front [9].

The correction functional for Eq. (1) can be approximately expressed and put A=u, as follows:

$$u_{n+1} = u_n + \int_0^t \lambda(\tau)\{u_{n\tau} + 2\tilde{u}_n \tilde{u}_{nx} - \frac{\sqrt{\tilde{u}_n}}{2}\tilde{u}_{nxx} - \frac{\tilde{u}_{nx}^{\,2}}{u\sqrt{\tilde{u}_n}}\} d\tau \qquad (3)$$

Where λ is a general lagrange multiplier, $\tilde{u}_n$ is considered as restricted variations and $\delta\tilde{u}_n$ is considered as a restricted variation.

Making the above correction functional stationary and noticing that $\delta\tilde{u}_n$ = 0, we obtain:

$$\delta\tilde{u}_{n+1} = \delta\tilde{u}_n + \int_0^t \delta\lambda(\tau)\{u_{n\tau} + 2\tilde{u}_n\tilde{u}_{nx} + \frac{\sqrt{\tilde{u}_n}}{2}\tilde{u}_{nxx} - \frac{(\tilde{u}_{nx})^2}{u\sqrt{\tilde{u}_n}}\}d\tau \qquad (4)$$

or

$$0 = \delta u_n + \int_0^t \delta\lambda(\tau)\{u_{n\tau}\}d\tau \qquad (5)$$

$$0 = \delta u_n + \lambda(\tau)\delta u_n + \int_0^t \delta\lambda'(\tau)u_n d\tau, \qquad (6)$$

Which produces the stationary conditions:

$$\lambda'(t) = 0 \qquad (7a)$$

$$1 + \lambda(t)|_{\tau=t} = 0 \qquad (7b)$$

Where Eq. (7a) is called lagrange- Euler equation and Eq. (7b) natural boundary condition.

The lagrange multiplier, there fore, can be identified as $\lambda = -1$ and the following variational iteration formula can be obtained:

$$u_{n+1} = u_n - \int_0^t \{u_{n\tau} + 2u_n u_{nx} - \frac{\sqrt{u_n}}{2}u_{nxx} - \frac{1}{u\sqrt{u_n}}(u_{nx})^2\}d\tau \qquad (8)$$

by the iteration formula (8), we can obtain the other components as:

$$u_0(x,t) = u(x,0)$$
$$u_1(x,t) = 3\tanh^2(\sqrt{3}x) + t(-36\sqrt{3}\sec h^2(\sqrt{3}x)\tanh^3(\sqrt{3}x)$$
$$+9\sqrt{3}\sec h^4(\sqrt{3}x) + \frac{\sqrt{3}}{2}\tanh(\sqrt{3}x)(18\sec h^4(\sqrt{3}x)$$
$$-36\sec h^2(\sqrt{3}x)\tanh^2(\sqrt{3}x)))$$

In the same manner the rest of components of the iteration formulas (8).

3. APPLYING PERTURBATION METHOD

Foam drainage equation (1) can be written in the form and put A=u, as follows:

$$u_t + (u^2 - \frac{\sqrt{u}}{2}u_x)_x = 0 \qquad (9)$$

we introduce the wave variable $\xi = x - ct$.

Where ξ is called the wave variable, and c is the wave speed .

By using the wave variable can be converted in to ODE.

$$u_t = -cw_\xi = -cw', \qquad (10)$$
$$u_x = w_\xi = w',$$

Where $' = \frac{d(.)}{d\xi}$. substituting from eq.(10) in eq.(9) given

$$-cw_\xi + (w^2 - \frac{\sqrt{w}}{2}w_\xi)_\xi = 0, \qquad (11)$$

Integrating eq.(11) with respect to ξ, and considering the constant of integration zero with $w(0)=0,$, we find

$$-cw+w^{2}-\frac{\sqrt{w}}{2}w_{\xi}=0, \qquad (12)$$

We next use the transformation $w=v^{2},\quad w'=2vv',$ that will convert eq.(12) to

$$-cv^{2}+v^{4}-v^{2}v'=0,$$

or equivalently

$$-c+v^{2}-v'=0, \qquad (13)$$

Assume that the solution of eq.(13) can be Taylor expanded in $\in$ then, we have

$$v=v_{0}+\in v_{1}+\in^{2}v_{2}+O(\in^{3}), \qquad (14)$$

for v_{0},v_{1},v_{2} to be determined where in order to satisfy the initial condition $v(0)=0$, we will have

$$v_{0}(0)=v_{1}(0)=v_{2}(0)=........=0, \qquad (15)$$

Substituting eq.(14), and eq.(15) in to eq.(13) gives

$$\begin{aligned}-c+v_{0}^{2}+\in v_{0}v_{1}+\in^{2}v_{0}v_{2}+............\\ +\in v_{0}v_{1}+\in^{2}v_{1}^{2}+\in^{3}v_{1}v_{2}+...\\ +\in^{2}v_{0}v_{2}+\in^{3}v_{1}v_{2}+\in^{4}v_{0}^{2}+...\\ -v_{0}'-\in v_{1}'-\in^{2}v_{2}'+.............=0,\end{aligned}$$

and we can collect powers of $\in$:

$$O(\in^0): \quad -c+v_0^2-v_0'=0,$$
$$O(\in^1): \quad 2v_0v_1-v_1'=0,$$
$$O(\in^2): \quad 2v_0v_2+v_1^2-v_2'=0,$$
$$O(\in^3): \quad ....$$

Now we simply solve at each order, applying the boundary conditions as we obtain the following solutions for v_0, v_1, v_2, then

$$v_0=-\sqrt{c}\tanh(\sqrt{c}\xi),$$
$$v_1=0,$$
$$v_2=0,$$

This in turn gives

$$v(x,t)=-\sqrt{c}\tanh(\sqrt{c}(\mathrm{x}-\mathrm{ct})),$$

And by noting that $w=v^2$, and eqs.(10) we find

$$u(x,t)=c\tanh^2(\sqrt{c}(\mathrm{x}-\mathrm{ct})),$$

4. NUMERICAL RESULTS AND COMPARING THESE METHOD

Tables [(1), (2), (3)] investigated comparison between errors of by ADM , VIM, and PM at t=0.1, 0.01 ,0.001.

TABLE (1) COMPARISON BETWEEN ERRORS OF ADM, VIM, PM AND PM FOR EQ.(1) , WHEN C=3, AND T=0.1 .

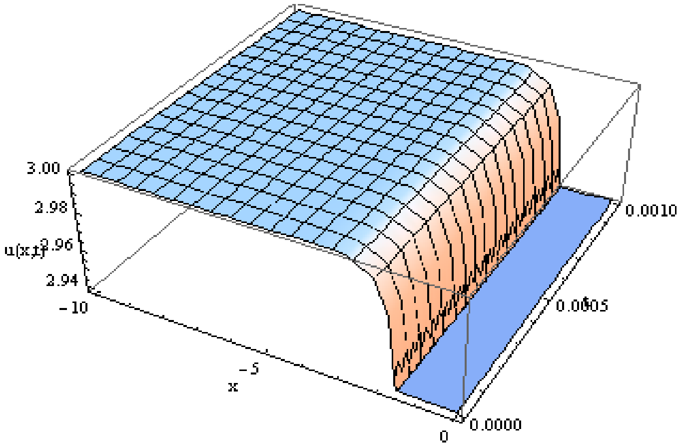
x_i	$\lvert u_{Exact} - u_{ADM} \rvert$ [8]	$\lvert u_{Exact} - u_{VIM} \rvert$	$\lvert u_{Exact} - u_{PM} \rvert$
-10	1.42109e-14	4.44089e-15	0.0
-8	1.40941e-11	4.34497e-12	0.0
-6	1.43842e-08	4.43491e-09	0.0
-4	0.0000146727	4.52651e-06	0.0
-2	0.0064218	0.00459239	0.0
0	3.71367	0.683741	0.0

TABLE (2) COMPARISON BETWEEN ERRORS OF ADM, VIM, PM AND PM FOR EQ.(1) WHEN C=3, , AND T=0.01.

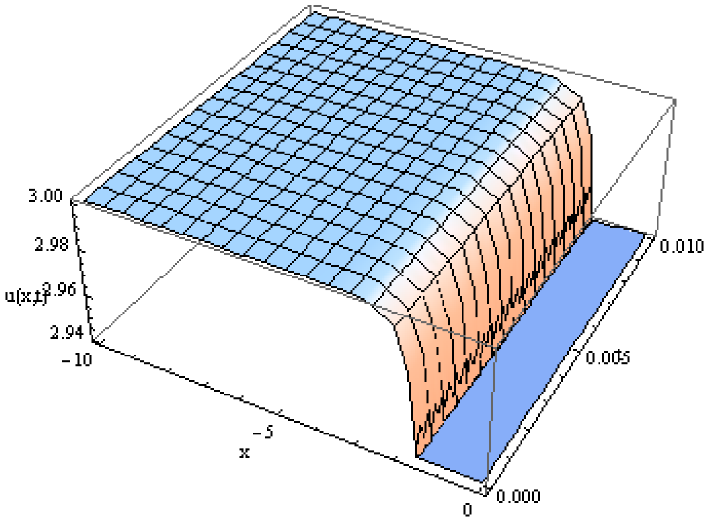
x_i	$\lvert u_{Exact} - u_{ADM} \rvert$ [8]	$\lvert u_{Exact} - u_{VIM} \rvert$	$\lvert u_{Exact} - u_{PM} \rvert$
-10	1.77636e-15	4.44089e-16	0.0
-8	1.86962e-12	5.77316e-14	0.0
-6	1.9087e-09	5.88871e-11	0.0
-4	1.94811e-06	6.0103e-08	0.0
-2	0.00197296	6.08817e-05	0.0
0	0.00051592	0.00808544	0.0

TABLE (3) COMPARISON BETWEEN ERRORS OF ADM, VIM, AND PM FOR EQ.(1) WHEN C=3, , AND T=0.001.

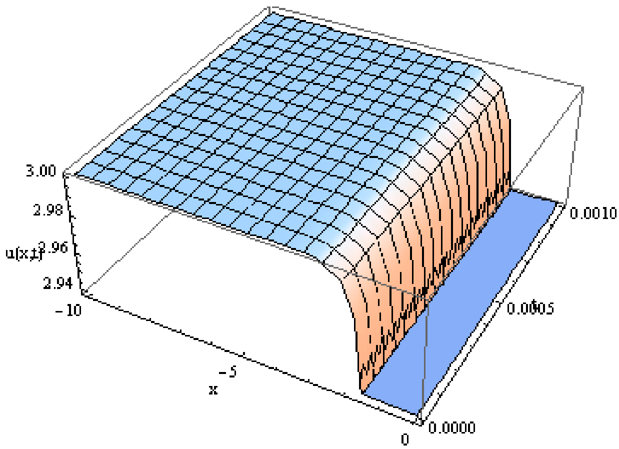
x_i	$\lvert u_{Exact} - u_{ADM} \rvert$ [8]	$\lvert u_{Exact} - u_{VIM} \rvert$	$\lvert u_{Exact} - u_{PM} \rvert$
-10	4.44089e-16	4.44089e-16	0.0
-8	2.24265e-13	4.44089e-16	0.0
-6	2.29754e-10	6.07514e-13	0.0
-4	2.34498e-07	6.1988e-10	0.0
-2	0.000236656	6.27765e-7	0.0
0	5.2479e-08	0.00809985	0.0



The exact solution for eq.(1)



The VIM solution for eq.(1)



The PM solution for eq.(1)

Fig.1. Comparison between the exact solution, the VIM solution and PM solution for eq.(1) when c=3.

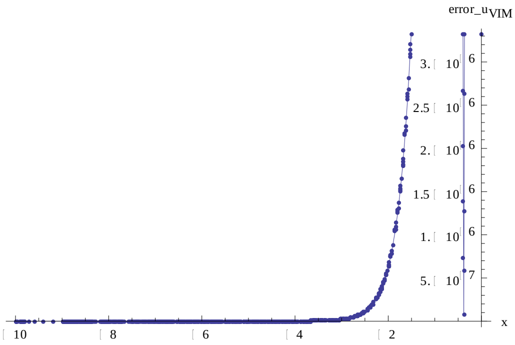


Fig2. The absolute error of (VIM) for eq.(1) at when c=3 and t=0.001.

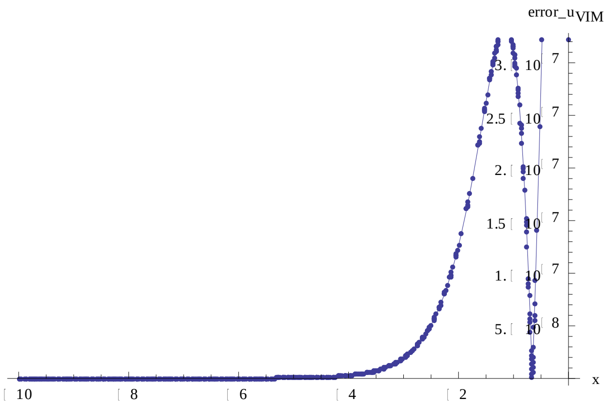


Fig3. The absolute error of (VIM) for eq.(1) at when c=1 and t=0.001.

5. RESULTS AND DISCUSSION.

In this paper, the first one is to implement the standard VIM, and PM, to handle the foam drainage equation, and to emphasize the strength of these methods in handling nonlinear differential equations.

We aim in the second goal to conduct a comparative study between the proposed methods to emphasize the strength of the three methods in handling nonlinear problems.

We was obtained exact solutions by using the perturbation method. However approximations solutions of high degree of accuracy were obtained by using VIM, the comparison was by evaluating the errors between exact solutions and approximate solutions, ADM results in tables(1,2,3), initially the computational error was smaller at t=0.1 where it was $|error\_u| \leq 10^{-15}$ while it was the best at t=0.001 where it was $|error\_u| \leq 10^{-16}$.

The results revealed that errors obtained by using the VIM $10^{-15} \leq |error\_u| \leq 10^{-01}$ are more efficient than that errors obtained by using the ADM $10^{-14} \leq |error\_u_{ADM}| \leq 3.7$ but the perturbation method provided the exact solution to assure its power.

6. CONCLUSION

This paper dealt two methods for solving the Foam Drainage Equation, an explicit analytical solution obtained by Variational iteration method wich is a powerful mathematical tool in dealing with non linear equation, and comparison of results in tables(1) (2)and (3),but we get the exact solution by perturbation method.

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