RESOLUTION OF THE MANSURIPUR'S PARADOX

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ABSTRACT

In 2012 Masud Mansuripur claims that the Lorentz force law is incompatible with special relativity in an article appeared in Physical Review Letters. In this paper I discuss the paradox on which this claim is based. The resolution depends on whether one assumes a Gilbert model for the magnetic dipole (separated monopoles) or the standard Ampere model (a current loop). The former case was treated some years ago and the latter will be treated in this paper. It constitutes an interesting manifestation of hidden momentum.

***Keywords*:**Lorentz law; Lorentz transformation; magnetic dipole; Mansuripur's Paradox; Hidden Momentum; Einstein-Laub Force Law.

1. INTRODUCTION

On May 7, 2012, a remarkable article appeared in *Physical Review Letters* [1]. The author, Masud Mansuripur, claimed to offer an incontrovertible theoretical evidence of the incompatibility of the Lorentz force law with the fundamental tenets of special relativity, and concluded that the Lorentz law must be abandoned. The Lorentz law [$\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$] determines the force $\boldsymbol{F}$ on a charge q moving with velocity $\boldsymbol{v}$ through electric field $\boldsymbol{E}$ and magnetic field $\boldsymbol{B}$. Together with Maxwell's equations, it is the foundation on which all of classical electrodynamics rests. If it is incorrect, 150 years of theoretical physics is in serious risk. In response to such a provocative proposal, *Science* [2] published a full-page commentary, and within days several rebuttals were posted [3]. Critics pointed out that since the Lorentz force law can be formulated in a manifestly covariant formulation of electrodynamics, it is guaranteed to be consistent with special relativity, and some of them identified the specific source of Mansuripur's error [4].

Mansuripur's argument is based on a paradox that was explored by Victor Namias [5] many years ago. In Section **2** I introduce Mansuripur's version of the paradox, in simplified form, and In Section **3** I explain Namias's resolution which based on a Gilbert model of the dipole. In Section **4** I discuss the physical nature of hidden momentum; a phenomenon involved in the resolution based on Amperian model of the dipole, which I deal with in section **5**. Mansuripur himself treated the dipole as the point limit of a magnetized object, so in Section **6** I repeat the calculations in that context for both models, and confirm the earlier results. In Section **7** I discuss the Einstein-Laub force law, which Mansuripur proposed as a replacement for the Lorentz law, and in Section **8** I offer some comments and conclusion.

2. THE PARADOX

Suppose that there are an ideal magnetic dipole $\boldsymbol{m}$ and a point charge q, both at rest inthe proper frame S'. The torque on $\boldsymbol{m}$ is obviously zero, because there is no magnetic field. Now examine the same configuration in the lab frame S, which moves at constant speed $\boldsymbol{v}$with respect to S'. In S the moving point charge generates electric and magnetic fields in the direction ($\parallel$) and perpendicular ($\perp$) to $\boldsymbol{v}$according to Lorentz transformation [6]:

$$\boldsymbol{B}_{\perp} = \gamma\left(\frac{\boldsymbol{v}\times\boldsymbol{E}'}{c^2}\right), \boldsymbol{E}_{\perp} = \gamma\boldsymbol{E}'_{\perp}, \qquad \boldsymbol{B}_{\parallel} = 0, \boldsymbol{E}_{\parallel} = \boldsymbol{E}'_{\parallel} whe \quad \gamma = 1\Big/\sqrt{1-\frac{v^2}{c^2}} \quad (1)$$

and the moving magnetic dipole acquires an electric dipole moment [7]:

$$\boldsymbol{p} \qquad = \frac{\boldsymbol{v} \times \boldsymbol{m}}{c^2} (2)$$

The torque on the dipole is [8]:

$$\boldsymbol{\tau} = (\boldsymbol{m} \times \boldsymbol{B}) + (\boldsymbol{p} \times \boldsymbol{E}) = \frac{1}{c^2} [\boldsymbol{m} \times (\boldsymbol{v} \times \boldsymbol{E}) + \times \boldsymbol{E}] (3)$$

But

$$\boldsymbol{m} \times (\boldsymbol{v} \times \boldsymbol{E}) = \boldsymbol{v} \times (\boldsymbol{m} \times \boldsymbol{E}) - (\boldsymbol{v} \times \boldsymbol{m}) \times \boldsymbol{E} (4)$$

so

$$\boldsymbol{\tau} = \frac{\boldsymbol{v} \times (\boldsymbol{m} \times \boldsymbol{E})}{c^2} (5)$$

The torque is zero in one inertial frame, but non-zero in the other! Mansuripur concludes that the Lorentz force law on which Eq. 3 is predicated is inconsistent with special relativity.

3. NAMIAS'S RESOLUTION

Victor Namias was resolved this paradox few years ago by showing that the standard torque formula (Eq. 3) is applied to dipoles at rest, but they do not hold, in general, for dipoles in motion [5].

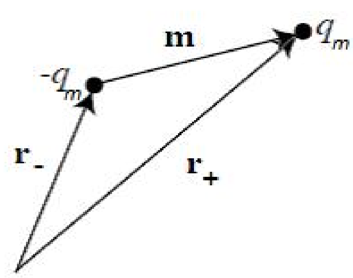

Figure 1. Gilbert magnetic dipole.

Suppose that we model the magnetic dipole as separated monopoles (Fig. 1).

$$\boldsymbol{m} = q_m(\boldsymbol{r}_+ - \boldsymbol{r}_-)(6)$$

The Lorentz force law for a magnetic monopole q_m reads [8]:

$$\boldsymbol{F} = q_m\left(\boldsymbol{B} - \frac{\boldsymbol{v} \times \boldsymbol{E}}{c^2}\right)(7)$$

so the torque on a moving dipole:

$$\boldsymbol{\tau} = (\boldsymbol{r}_+ \times \boldsymbol{F}_+) + (\boldsymbol{r}_- \times \boldsymbol{F}_-) = \boldsymbol{m} \times \boldsymbol{B} + \frac{1}{c^2}\boldsymbol{m} \times (\boldsymbol{v} \times \boldsymbol{E})(8)$$

Using Eq. 4 and Eq. 2 we get:

$$\boldsymbol{\tau} = (\boldsymbol{m} \times \boldsymbol{B}) + (\boldsymbol{p} \times \boldsymbol{E}) - \frac{\boldsymbol{v} \times (\boldsymbol{m} \times \boldsymbol{E})}{c^2}(9)$$

There is a third term, missing in Eq. 3, which exactly cancels the offending torque; the net torque is zero in both frames.

4. HIDDEN MOMENTUM

A current loop with magnetic dipole $\boldsymbol{m}$ in an electric field $\boldsymbol{E}$ carries hidden momentum equal to [8]:

$$\boldsymbol{p}_h = \frac{\boldsymbol{m} \times \boldsymbol{E}}{c^2}(10)$$

The term "hidden momentum" was coined by Shockley [9]. The phenomenon itself was first studied in the context of static electromagnetic systems with nonzero field momentum $\left(\boldsymbol{p}_f = \epsilon_0 \int (\boldsymbol{B} \times \boldsymbol{E})\, d^3\boldsymbol{r}\right)$. In such configurations the hidden momentum exactly cancels the field momentum $(\boldsymbol{p}_h = -\boldsymbol{p}_f)$leaving a total of zero, as required by the centre of energy theorem (*if the centre of energy of a closed system is at rest, then its total momentum is zero* [10]). This has created the impression that hidden momentum is something artificial and ad hoc invented simply to rescue an abstract theorem. Mansuripur variously calls hidden momentum "an absurdity" [11], but nothing could be farther from the truth. Hidden momentum is perfectly ordinary relativistic mechanical momentum, it occurs in systems with internally moving parts, such as current-carrying loops, and it is "hidden" only in the sense that it is not associated with motion of the object as a whole. A Gilbert dipole in an electric field, having no moving parts, harbours no hidden momentum and the fields carry no compensating momentum.

5. AMPERIAN MODEL

Namias believed that his formula (Eq. 9) is applied just as well to an Ampere dipole as it does to a Gilbert dipole. But e was mistaken.

An Ampere dipole in an electric field carries hidden momentum. In the proper frame S', the hidden momentum is given by Eq. 10 which is constant in time, so there is no associated force. There is no associated torque too because the hidden angular momentum ($\boldsymbol{L}_h = \boldsymbol{r} \times \boldsymbol{p}_h$) is constant in time.

Because $\boldsymbol{p}_h$ is perpendicular to $\boldsymbol{v}$, and transverse components are unaffected by Lorentz transformations, the hidden momentum in lab frame S is the same but the hidden angular momentum is not constant, because r is changing.

$$\frac{d\boldsymbol{L}_h}{d} = \boldsymbol{v} \times \boldsymbol{p}_h = \frac{\boldsymbol{v} \times (\boldsymbol{m} \times \boldsymbol{E})}{c^2} \quad (11)$$

This increase in angular momentum requires a torque:

$$\boldsymbol{\tau} = \frac{\boldsymbol{v} \times (\boldsymbol{m} \times \boldsymbol{E})}{c^2} \quad (12)$$

and this is precisely what have been found in Eq. 5, but it drives the increasing hidden angular momentum which is not associated with any actual rotation of the object (this is certainly not the first time such issues have arisen [12]).

In the Gilbert model there is an extra term in the torque formula (Eq. 9); the total torque is zero, there is no hidden angular momentum, and nothing rotates. In the Ampere model there is no third term in the torque formula; the torque is not zero, and drives the increasing hidden angular momentum (Eq. 12), but still nothing rotates.

6. MAGNETIZED MATERIALS

It is of interest to see how this plays out in Mansuripur's formulation of the problem. He treats the dipole as magnetized medium, and calculates the torque directly from the Lorentz force law, without invoking $(\boldsymbol{m} \times \boldsymbol{B})$ or $(\boldsymbol{p} \times \boldsymbol{E})$. In the proper frame S', for an ideal magnetic dipole $\boldsymbol{m} = m_0\hat{\boldsymbol{x}}$ at $(0; 0; d)$, and a point charge q at the origin, he takes [1]:

$$\boldsymbol{M}'(x',y',z',t') = m_0\delta(x')\delta(y')\delta(z'-d)\hat{\boldsymbol{x}} \quad (13)$$

$\boldsymbol{M}$ and $\boldsymbol{P}$ constitute an antisymmetric second-rank tensor:

$$P^{\mu} = \begin{bmatrix} 0 & cP_x & cP_y & cP_z \\ -cP_x & 0 & -M_z & M_y \\ -cP_y & M_z & 0 & -M_x \\ -cP_z & -M_y & M_x & 0 \end{bmatrix} \quad (14)$$

and the transformation rule for motion in the z direction is:

$$P_z = P_z{}', \qquad P_y = \gamma\left(P_y{}' - \frac{v}{c^2}M_x{}'\right), \qquad P_x = \gamma\left(P_x{}' + \frac{v}{c^2}M_y{}'\right)$$

$$M_z = M_z{}', \qquad M_y = \gamma\left(M_y{}' + vP_x{}'\right), \qquad M_x = \gamma\left(M_x{}' - vP_y{}'\right)$$

In the present case, then, the polarization and magnetization in the lab frame S are:

$$\boldsymbol{M}(x,y,z,t) = m_0\delta(x)\delta(y)\delta\left(z - v \ - \frac{d}{\gamma}\right)\hat{\boldsymbol{x}} \quad (15)$$

$$\boldsymbol{P}(x,y,z,t) = \frac{m_0 v}{c^2}\delta(x)\delta(y)\delta\left(z - v \ - \frac{d}{\gamma}\right)\hat{\boldsymbol{y}} \quad (16)$$

According to the Lorentz law, the force density is [8]:

$$\boldsymbol{f} = \rho\boldsymbol{E} + \boldsymbol{J} \times \boldsymbol{B} \quad (17)$$

The charge density here ρ is the bound charge density and the current density $\boldsymbol{J}$is the sum of the polarization current and the bound current density [8]:

$$\rho = -\boldsymbol{\nabla} \cdot \boldsymbol{P}, \qquad \boldsymbol{J} = \frac{\partial \boldsymbol{P}}{\partial} + \boldsymbol{\nabla} \times \boldsymbol{M} \qquad (18)$$

The fields $\boldsymbol{E}$and $\boldsymbol{B}$ can be calculated using the transformation rule in Eq. 1 for motion in the z direction:

$$\boldsymbol{E}(x, y, z, \boldsymbol{t}) = \left(\frac{1}{4\pi\varepsilon_0}\right)\frac{q}{R^3}(x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + (z - v\ )\hat{\boldsymbol{z}})(19)$$

$$\boldsymbol{B}(x, y, z, t) = \left(\frac{1}{4\pi\varepsilon_0}\right)\frac{q}{c^2R^3}(-y\hat{\boldsymbol{x}} + x\hat{\boldsymbol{y}})(20)$$

where $R = \sqrt{x^2 + y^2 + \gamma^2(z - v\ )^2}$. Putting all in Eq. 17, we obtain:

$$\boldsymbol{f} = -\left(\frac{1}{4\pi\varepsilon_0}\right)\frac{m_0 q}{c^2R^3}\delta(x)\frac{\partial}{\partial}\delta(y)\delta\left(z - v\ \ - \frac{d}{\gamma}\right)\hat{\boldsymbol{z}}(21)$$

The net force on the dipole is:

$$\boldsymbol{F} = \int \boldsymbol{f}\, d\ \ d\ \ d\ \ = \frac{m_0 q}{4\pi\varepsilon_0 c^2}\left[\frac{d}{d}\left(\frac{1}{(y^2 + d^2)^{3/2}}\right)\right]_{y=0}\hat{\boldsymbol{z}} = 0(22)$$

Meanwhile, the torque density is:

$$\boldsymbol{\mathcal{T}} = \boldsymbol{r} \times \boldsymbol{f} = -\left(\frac{1}{4\pi\varepsilon_0}\right)\frac{m_0 q}{c^2R^2}\delta(x)\frac{\partial}{\partial}\delta(y)\delta\left(z - v\ \ - \frac{d}{\gamma}\right)\hat{\boldsymbol{x}}(23)$$

so the net torque on the dipole is:

$$\boldsymbol{\tau} = \int \boldsymbol{\mathcal{T}}\, d \quad d \quad d \quad = \frac{m_0 q}{4\pi\varepsilon_0 c^2}\left[\frac{d}{d}\left(\frac{y}{(y^2 + d^2)^{3/2}}\right)\right]_{y=0} \hat{\boldsymbol{x}}$$

$$= \left(\frac{1}{4\pi\varepsilon_0}\right)\frac{m_0 q}{c^2 d^2}\,\hat{\boldsymbol{x}} \quad (24)$$

It is easy to check that this is the torque required to account for the increase in hidden angular momentum (Eq. 12):

$$\boldsymbol{\tau} = \frac{\boldsymbol{v} \times (\boldsymbol{m} \times \boldsymbol{E})}{c^2} = \left(\frac{1}{4\pi\varepsilon_0}\right)\frac{m_0 q}{c^2 d^2}\,\hat{\boldsymbol{x}} \quad (25)$$

What if we apply Mansuripur's calculation for a dipole made out of magnetic monopoles? There is some dispute as to the correct form of the Lorentz force law for magnetic monopoles in the presence of polarizable and magnetizable materials, but when the polarization or magnetization is itself due to monopoles, as here; the charge density and current density are [8]:

$$\rho_m = -\boldsymbol{\nabla} \cdot \boldsymbol{M}, \qquad \boldsymbol{J}_m = \frac{\partial \boldsymbol{M}}{\partial} - c^2 \boldsymbol{\nabla} \times \boldsymbol{P} \quad (26)$$

so the force density on the magnetic dipole is:

$$\boldsymbol{f} = \rho_m \boldsymbol{B} - \frac{\boldsymbol{J}_m \times \boldsymbol{E}}{c^2} = -(\boldsymbol{\nabla} \cdot \boldsymbol{M})\boldsymbol{B} + (\boldsymbol{\nabla} \times \boldsymbol{P}) \times \boldsymbol{E} - \frac{1}{c^2}\left(\frac{\partial \boldsymbol{M}}{\partial} \times \boldsymbol{E}\right) = 0 \quad (27)$$

The total force is again zero, but this time so too is the torque density, and hence the total torque. As before, the torque is zero in the Gilbert model and there is no hidden angular momentum.

7. EINSTEIN-LAUB FORCE LAW

Having concluded that the Lorentz force law is unacceptable, Mansuripur proposes to replace Lorentz force law with an expression based on the Einstein-Laub law [13]:

$$\boldsymbol{f}_E = (\boldsymbol{P}\cdot\nabla)\boldsymbol{E} + \frac{\partial \boldsymbol{P}}{\partial\left(\frac{\partial \boldsymbol{M}}{\partial}\times\boldsymbol{E}\right)t}\times\mu_0\boldsymbol{H} + (\boldsymbol{M}\cdot\nabla)\mu_0\boldsymbol{H} - \frac{1}{c^2} \quad (28)$$

In our case Einstein-Laub law gives:

$$\boldsymbol{f}_E = \left(\frac{1}{4\pi\varepsilon_0}\right)\frac{m_0 q}{c^2R^3}\delta(x)\delta(y)\left[2\delta\left(z-v-\frac{d}{\gamma}\right) - (z-v)\frac{\partial}{\partial}\delta\left(z-v-\frac{d}{\gamma}\right)\right]\hat{\boldsymbol{y}} \quad (29)$$

The total force on the dipole still vanishes:

$$\boldsymbol{F}_E = \frac{m_0 q}{4\pi\varepsilon_0 c^2}\left[\frac{2}{d^3} + \frac{1}{\gamma^3}\frac{d}{d}\left(\frac{1}{(z-v)^2}\right)\right]_{z-v=d/\gamma}\hat{\boldsymbol{y}} = 0 \quad (30)$$

The torque density should be:

$$\boldsymbol{\mathcal{T}}_E = -\left(\frac{1}{4\pi\varepsilon_0}\right)\frac{m_0 q}{c^2R^3}\delta(x)\delta(y)\left[2\delta\left(z-v-\frac{d}{\gamma}\right) - (z-v)\frac{\partial}{\partial}\delta\left(z-v-\frac{d}{\gamma}\right)\right]\hat{\boldsymbol{x}} \quad (31)$$

giving a total torque:

$$\boldsymbol{\tau}_E = -\frac{m_0 q}{4\pi\varepsilon_0 c^2}\left[\frac{2\left(v+\frac{d}{\gamma}\right)}{d^3} + \frac{1}{\gamma^3}\frac{d}{d}\left(\frac{z}{(z-v)^2}\right)\right]_{z-v=d/\gamma}\hat{\boldsymbol{x}} = -\left(\frac{1}{4\pi\varepsilon_0}\right)\frac{m_0 q}{c^2 d^2}\hat{\boldsymbol{x}} \quad (32)$$

It's not zero! In fact, it's minus the Lorentz torque (Eq. 24). But Mansuripur argues that, "To guarantee the conservation of angular momentum, Eq. 31 must be supplemented "

$$\boldsymbol{\mathcal{T}}'_E = \boldsymbol{\mathcal{T}}_E + (\boldsymbol{m} \times \boldsymbol{B}) + (\boldsymbol{p} \times \boldsymbol{E}) \quad (33)$$

In our case the extra terms are:

$$(\boldsymbol{m} \times \boldsymbol{B}) + (\boldsymbol{p} \times \boldsymbol{E}) = \left(\frac{1}{4\pi\varepsilon_0}\right)\frac{m_0 q}{c^2 d^2}\delta(x)\delta(y)\delta\left(z - v \quad - \frac{d}{\gamma}\right)\hat{\boldsymbol{x}} \quad (34)$$

and their contribution to the total torque is:

$$\int (\boldsymbol{m} \times \boldsymbol{B}) + (\boldsymbol{p} \times \boldsymbol{E}) d \quad d \quad d \quad = \left(\frac{1}{4\pi\varepsilon_0}\right)\frac{m_0 q}{c^2 d^2}\hat{\boldsymbol{x}} \quad (35)$$

which is just right to cancel Eq. 32, yielding a net torque of zero, which Mansuripur takes to be the correct answer.

8. COMMENTS AND CONCLUSION

What are we to make of Mansuripur's argument? In the first place, the Einstein-Laub force density was derived assuming that the medium is at rest [13], which in this case is not. More important, the magnetization terms in Einstein-Laub force law implicitly assume a Gilbert model for the magnetic dipole, but Mansuripur is quite explicit in writing that the magnetic dipole he has in mind is "a small, charge neutral loop of current," which is to say, an Ampere dipole. There may be some contexts in which the Einstein-Laub force law is valid and useful, but this is not one of them. The resolution of Mansuripur's paradox for both models of the magnetic dipole; Gilbert model, and Ampere model is:

In Gilbert dipole (made from magnetic monopoles), the third term in Namias's formula (Eq. 9) supplies the missing torque. In Mansuripur's formulation (using a polarizable medium), it comes from a correct accounting of the charge density and current density (Eq. 26). The net torque is zero in the lab frame, just as it is in the proper frame.

In Ampere dipole (an electric current loop), the third term in Namias's equation is not exist, and the torque on the dipole is not zero. It is, however, just right to account for the increasing hidden angular momentum in the dipole.In either model, the Lorentz force law is entirely consistent with special relativity.

REFERENCES

[1] M. Mansuripur, "Trouble with the Lorentz Law of Force: Incompatibility with Special Relativity and Momentum Conservation", Phys. Rev. Lett. 108, 193901 (2012).
[2] A. Cho, "Textbook Electrodynamics May Contradict Relativity", Science 336, 404 (2012).
[3] T. M. Boyer, "Examples and Comments Related to Relativity Controversies", Am. J. Phys. 80, 962 (2012); A. L. Kholmetskii, O. V. Missevitch, and T. Yarman, "Torque on a Moving Electric/Magnetic Dipole", Prog. Elecromagnetics Research B, 45, 83 (2012).
[4] D. A. T. Vanzella, "Comment on Trouble with the Lorentz Law of Force: Incompatibility with Special Relativity and Momentum Conservation", Phys. Rev. Lett. 110, 089401 (2013); and, under the same title, S. M. Barnett, Phys. Rev. Lett. 110, 089402 (2013); P. L. Saldanha, Phys. Rev. Lett. 110, 089403 (2013); M. Khorrami, Phys. Rev. Lett. 110, 089404 (2013).
[5] V. Namias, "Electrodynamics of Moving Dipoles: The Case of the Missing Torque", Am. J. Phys. 57, 171 (1989)
[6] R. Leighton, Principles of Modern Physics, New York, McGraw-Hill, (1957).
[7] V. Hnizdo, "Magnetic Dipole Moment of a Moving Electric Dipole", Am. J. Phys. 80, 645 (2012).
[8] D. J. Griffiths, Introduction to Electrodynamics, Boston, Pearson, (2013).
[9] W. Shockley and R. P. James, "Try simplest cases: discovery of hidden momentum forces on magnetic currents", Phys. Rev. Lett. 18, 876 (1967).
[10] S. Coleman and J. H. Van Vleck, "Origin of Hidden Momentum Forces on Magnets", Phys. Rev. 171, 1370 (1968)

[11] M. Mansuripur, "Resolution of the Abraham-Minkowski controversy", Opt. Commun. 283, 1997 (2010)
[12] D. G. Jensen, "The paradox of the L-shaped object", Am. J. Phys. 57, 553 (1989).
[13] A. Einstein and J. Laub, "Uber die im elektromagnetischen Felde auf ruhende Korper ausgeubten ponderomotorischen Krafte", Ann. Phys. 26, 541 (1908); English translation in The Collected Papers of Albert Einstein, Vol. 2, Princeton, Princeton University Press, (1989).