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Structure of Open Self-Maps on Topological Spaces That Satisfy a Separation Axiom

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Abstract:
Let X be any set and $f: X \to X$ be a map. In this paper, the following questions are addressed: Can we define a non-discrete topology on X with respect to which f is an open map? When can we endow X with a non-discrete topology such that the map f is open and X satisfies one of the separation axioms? We answer some of these questions in terms of the orbit structure of the map concerned.

Keywords: Orbit, Open map, Topological space, Separation axiom.
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بنية الدوال المفتوحة المعرفة علي الفضاءات التبولوجية التي تحقق احدى مسلمات الفصل

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الملخص:
اذا كانت f دالة معرفة على مجموعة X، في هذا البحث تم دراسة الحالات التي يمكن فيها تعريف تبولوجيا علي X بحيث تكون الدالة f مفتوحة و المجموعة X تحقق احدى مسلمات الفصل، وذلك من خلال معرفة البنية المدارية للدالة المعطاة f.

الكلمات المفتاحية: مدار، دالة مفتوحة، فضاء تبولوجي، بنية مدارية، مسلمة فصل.

1. Introduction

Let f be a self-map on a non-empty set X. Ellis in [2] asked the following question: is there a non-trivial topology on X such that f is continuous? Groote and Vries in [6] gave the complete answer to this question. In [4] and [5], the following questions were answered: If $f: X \to X$ is a function, when can we put a compact and Hausdorff topology on X that makes f continuous (or upper semi-continuous)?. There were some answers to similar questions with different topological properties on the set X; see for example [1], [6] and [7].
In the present paper, we study the following questions: Can we define a topology on X with respect to which f is an open map? Can we define a topology on X with respect to which f is an open map and X is a T_0 space, a T_1 space, a regular space or a normal space? The answer to most of these questions was given in terms of the orbit structure of f.

2. Preliminaries

To state our theorems, we give some preliminary definitions. Throughout this paper, by a non-discrete topology, we mean a topology that is neither discrete nor trivial. A topological space is normal if every pair of disjoint closed sets can be separated by disjoint open sets. A topological space is regular if any closed set F and any point x not contained in F can be separated by disjoint open sets.

Definition 2.1 [4] Let $g: X \to X$ be a map and let $\sim$ be an equivalence relation on X defined as:

$$x_1 \sim x_2 \Leftrightarrow \exists\, n, k \in \mathbb{N},\ g^n(x_1) = g^k(x_2).$$

The equivalence classes of the relation $\sim$ are the orbits of g.

Definition 2.2 [4] Let $f: X \to X$ be a map and O be an orbit of the map f.

1) O is called an n-cycle, for some $n \in \mathbb{N}$, if there are distinct points $\{y_0, \dots, y_{n-1}\}$ in O such that $f(y_k) = y_{k+1}$, where k is taken modulo n.

2) O is called a $\mathbb{Z}$-orbit if there are distinct points $\{y_k : k \in \mathbb{Z}\} \subseteq O$ such that $f(y_k) = y_{k+1}$ for all $k \in \mathbb{Z}$.

3) O is called a $\mathbb{N}$-orbit if there are distinct points $\{y_k : k \in \mathbb{N}\} \subseteq O$ such that $f(y_k) = y_{k+1}$ for all $k \in \mathbb{N}$.

Observation 2.1 [4] O is an $\mathbb{N}$ -orbit if it is neither a $\mathbb{Z}$-orbit nor an n -cycle.

Observation 2.2 [4] (1) The set $S = \{y_k : k \in L\}$ which determines that O is an n-cycle, a $\mathbb{Z}$-orbit or an $\mathbb{N}$-orbit of f is called a spine for O, where L is $\{0, \dots, n\}$, $\mathbb{Z}$ or $\mathbb{N}$.

(2) The set of all orbits of the map f is called the orbit structure of f.

Definition 2. 3 [4] Let O be an orbit of a map $f: X \to X$. O is called a simple orbit if $f|O$ is injective, in other words, the orbit O contains only a spine.

3. Open Self-Maps With One Simple Cycle

Lemma 3.1 Let X be a finite topological space and $f: X \to X$ be an open bijection that has only one simple n-cycle, then X has a pairwise disjoint basis $\{B_1, B_2, \dots, B_m\}$, where
1) $|B_i| = |B_j| = k$ for all i, j;
2) $m|n$ and $k|n$;
3) $k \leq |W|$ for any open set W.
Proof. Let U be a non-empty open subset of X with the property that $|U| \leq |W|$ for any open set W, so we have U, $f(U)$, $f^2(U), \dots,$ $f^{n-1}(U)$ are open sets with $|U| = |f(U)| = \dots = |f^{n-1}(U)| = k$ and

$$U \cup f(U) \cup \dots \cup f^{n-1}(U) = X$$

Obviously, any two sets in the collection $\{U, f(U), f^2(U), \dots, f^{n-1}(U)\}$ are either disjoint or equal; because if not, we will have an open set V with $|V| < |U|$, which is a contradiction.

Let m be the smallest integer for which U, $f(U)$, $f^2(U)$, $\dots$, $f^{m-1}(U)$ are disjoint with $U \cup f(U) \cup f^2(U) \cup \dots \cup f^{m-1}(U) = X$. Clearly, since $|U| = |f(U)| = \dots = |f^{m-1}(U)| = k$ and $|X| = n$, it follows immediately that $m|n$ and $k|n$.

Finally, we prove that the collection U, $f(U)$, $f^2(U)$, $\dots$, $f^{m-1}(U)$ is a basis for the topology of X. Suppose W is open and $x \in W$, so $x \in f^p(U)$ for some $0 \leq p \leq m-1$. If $f^p(U) \nsubseteq W$ then $W \cap f^p(U)$ is open with $|W \cap f^p(U)| < |U|$ which is a contradiction. Hence, $x \in f^p(U) \subseteq W$ and $U, f(U), f^2(U), \dots, f^{m-1}(U)$ is a basis.

Corollary 3.1 If $f: X \to X$ is an open bijection having only one simple q-cycle, where q is a prime number, then X is either the discrete space or the indiscrete space.

4. Structure of Open Self - Maps on *T₀* and *T₁* Spaces

Let X be an arbitrary set and $f: X \to X$ be a map. In this section, we will investigate the cases in which we can endow X with a topology such that f is open and X is a T_0 or a T_1 space. It is well-known that if X is finite T_1 space, the topology on X is the discrete topology. Now, we state our result.

Theorem 4.1 Let f be a map from an arbitrary set X to itself.

i) If X is an infinite set, then a non-discrete topology can always be defined on X so that f is an open map and X is a T_1 space.

ii) There is a non-discrete topology on X with respect to which f is an open map and X is a T_0 space if and only if the orbit structure of f is not only one simple n-cycle.

Proof. i) Let X be infinite and $f: X \to X$ be a map. We define a topology τ on X as the topology generated by the collection

$$\mathcal{S} = \{f^k(X), f^k(X - \{a\}): a \in X, k \in \mathbb{N}\}$$

It is simple to check that the map f is open. Since τ contains the co-finite topology, X is a T_1 space.

ii) If X is infinite, the proof follows from case (i) since every T_1 space is also T_0. If X is finite, we have two cases:

1) f has a non-simple cycle, so there is $b \in X$ such that $b \notin f(X)$. We define the excluded point topology on X, where b is the excluded point; so f is open and X is a T_0 space.

2) f has more than one simple cycle. If all cycles are 1-cycles, choose a point $a \in X$ and let τ_a be the particular point topology on X. Otherwise, choose an n-cycle O^* with $|n| > 1$ and define τ as the topology generated by the collection

$$\mathcal{S} = \left\{\{y\} \cup O^*: y \in X, y \notin O^*\right\} \cup \{\{b\}: b \in O^*\}$$

Clearly, f is open and X is a T_0 space.

Now, suppose that $f: X \to X$ is open with only one simple n-cycle, then there is a basis $\{V_1, V_2, \ldots, V_m\}$ that satisfies the conditions in Lemma 3.1. If $|V_i| = 1$, the topology is discrete and if $|V_i| > 1$ then for any two distinct points in V_i there is no open set that contains one of these points and not the other, so X is not a T_0 space.

5. Open Maps on Normal and Regular Spaces

In this section, the structure of open maps on regular and normal spaces will be studied. If $|X| \leq 2$, there is no non-discrete topology that makes X a regular space; while there is such a topology that makes X normal only when f has two 1-cycles.

Before we state our theorem, recall that if all open sets in X are clopen, then X is a regular and normal space.

Theorem 5.1 Let $f: X \to X$ be a surjective map and $|X| > 2$. There is always a non-discrete topology on X with respect to which f is an open map and X is a regular and normal space except when f has only a simple p-cycle, where p is prime.

Proof. From Corollary 3.1, if f has only a simple p-cycle, where p is prime, then no topology makes f an open map.

Now we prove the existence of a topology that makes X a regular and normal space and f an open map, we consider the following cases:

1) f is a bijection, so f has only simple orbits, we define the topology τ on X as follows:

i) If f has only one n-cycle $O = \{x_0, \ldots, x_{n-1}\}$ and n is not prime, let $d < n - 1$ be the largest divisor of n and let $E = \{x_{md}: 0 \leq m < d\}$, where md is taken modulo n. Let $\mathcal{S} = \{E, f(E), \ldots, f^{d-1}(E)\}$ and define a topology on X to be the topology generated by the sub-basis $\mathcal{S}$. Since the collection $\mathcal{S}$ is pairwise disjoint and n is not a prime number, this topology is a non-discrete topology.

ii) If the map f has only a $\mathbb{Z}$-orbit O, where $O = \{x_i: i \in \mathbb{Z}\}$, let $\tau = \{\phi,\ X,\ \{x_{2n}\},\ \{x_{2n+1}\} : n \in \mathbb{Z}\}$.

iii) If f has more than one simple orbit, choose an orbit O and let $\tau = \{\phi, X, O, O^C\}$.

Obviously, since in each of the cases above, all open sets are clopen, the space X is normal and regular. Also, it is simple to check that the map f is open.

2) f is not a bijection, choose an orbit of f with a spine S, let τ be the topology $\{\phi,\ X, S, S^C\}$. It follows immediately that the map f is open and the space X is normal and regular.

Corollary 5.1 Let $f: X \to X$ be any map on an arbitrary set X. There is always a non-discrete topology on X so that f is open and X is normal except when f has only a simple p-cycle, where p is prime.

Proof. If f is a surjection, the proof follows from Theorem 5.1. If f is not a surjection, let $\tau = \{\phi,\ f^k(X): k \in \mathbb{N}\}$, so f is open and X is normal.

Finally, from Corollary 5.1 and Corollary 3.1, we have the following.

Theorem 5.2 Let $f: X \to X$ be any map, a non-discrete topology can be defined on X such that f is an open map if and only if the orbit structure of f is not only a simple q-cycle, where q is a prime number.

6. Conclusion

The study has characterized the orbit structure of open self-maps on topological spaces, T_0 spaces, T_1 spaces and normal space. The case of surjective maps on regular spaces has been studied. Further studies and more cases can be investigated. The case of injective open maps on regular spaces can be studied. The case of a collection of self-maps on a topological space might be studied.

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