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SOME CONCEPTS ABOUT THE MULTIGRADED POLYNOMIAL SYSTEMS AND THE STRUCTURE OF NEWTON POLYTOPE

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Abstract:
Finding the solutions of a system of non – linear polynomial equations has received a lot of attention since ancient times. Recent active ongoing research related to solving such equations is on the construction and implemention of the method of sparse resultant. From the work of Emiris, an effective method for constructing sparse resultant matrices.This method relies the subdivision of the Minkowski sum of the Newton polytopes of polynomial systems, which generalizes the sparse elimination theory.In addition ,the mixed cells of the mixed subdivision can be used to compute the mixed volume of the Minikowski sum of Newton polytopes .

Keywords: Multigraded polynomial , Minkowski sum , Mixed volume and Newton polytope.

بعض المفاهيم حول أنظمة كثيرات الحدود متعددة الدرجات وتركيبة متعددة السطوح لنيوتن

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الملخص:
لقد حظي إيجاد الحلول لنظام متعددات الحدود غير الخطية بالكثير من الاهتمام منذ العصور القديمة. فالأبحاث الحديثة النشطة الجارية المتعلقة بحل مثل هذه المعادلات هي على بناء وتنفيذ طريقة محصلة سبارس المتناثرة. من عمل اميرس، طريقة فعالة لبناء المصفوفات ذات محصلة سبارس المتناثرة. تعتمد هذه الطريقة على التقسيم الفرعي لمجموع مينكوفسكي لنيوتن متعدد السطوح للأنظمة متعددات الحدود، والتي تعمم نظرية سبارس . بالإضافة إلى ذلك، يمكن استخدام الخلايا المختلطة للتقسيم المختلط لحساب الحجم المختلط لمجموع منكاوسكي لمحصلة نيوتن.

الكلمات المفتاحية: متعددات الحدود متعددة الدرجات , مجموع منكاوسكي , الحجم المختلط , محصلة نيوتن المتناثرة.

1. INTRODUCTION

A multivariable polynomial is linked to a polytope known as its Newton polytope. A polynomial is considered absolutely irreducible if its Newton polytope is indecomposable according to the Minkowski sum of polytopes. Two general methods for constructing indecomposable polytopes are presented, which yield numerous straightforward irreducibility criteria, including the famous Eisenstein criterion. The polynomials derived from these criteria can be defined over any field and maintain their absolute irreducibility even when their coefficients are altered arbitrarily within the field, provided that a specific set of coefficients remains nonzero.

This paper presents the definitions of multigraded polynomial system with some examples are presented [1]. Some examples from different types of multigraded bivariate polynomial system are illustrated [1]. The definitions of convex hull, Newton polytopes and Minikowski sum with some examples [2]. It is also shows the relationship between Newton polytopes of polynomial system and Minkowski sum of the given Newton polytopes [3].

2.MULTIGRADED SYSTEM

<u>Definition</u> 1 [1]:

A multigraded polynomial, a multi-homogeneous polynomial of type (l1, . . . , lr; h1, . . . , hr) is called multigraded if for each i = 1, . . . , r, either li = 1 or ki = 1.

<u>Note</u> :A multi-graded polynomial system F of type (l1, l2, . . . , lr; h1, h2, . . , hr) is an unmixed system of d + 1 generic multi-graded polynomials in d variables, where $\sum_{i=1}^{r} l_i$=d.

<u>Example 1:</u>

Consider the polynomials f_i= a_{i1} x^2y + a_{i2} x^2z + a_{i3} x^2t + a_{i4} xsy+ a_{i5} xsz + a_{i6} xst+ $a_{i7}y$s^2+ a_{i8} zs^2+ a_{i9} s^2 t for i=1,2,3,

Which is unmixed multigraded of type (1,2;2,1) with variable subset x,s and y,z,t, where the homogenizing variables are s and t in the respective block.

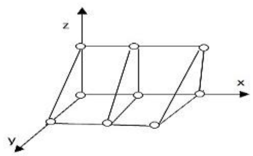


Figure (1):This is the Multi-graded system of type (1, 2; 2, 1)

Now,This figure depicts the support of a multi-graded system of type (1, 2; 2, 1), where variable blocks are {x} and {y, z}.

<u>Example 2:</u>

The following generic polynomial system in which all terms of degree 2 are present: *for i = 0, 1, 2* :

$$a_{i,00} + a_{i,01}y + a_{i,02}\, y^2 + a_{i,10}\, x + a_{i,11}\, xy + a_{i,20}\, x^2. \quad (1)$$

This system can be viewed as a multi-homogeneous polynomial of type (1, 1; 2, 2), in this case it is also multi-graded, but not full, because its support does not contain all the vertices that a system of type (1, 1; 2, 2) can have; in p

articular monomial x^2 y^2 is missing. After homogenizing variables, the system can be written as

$$a_{i,00}t^2z^2 + a_{i,01}ytz^2 + a_{i,02}\, y^2z^2 + a_{i,10}\, x\, t^2z + a_{i,11}\, xytz + a_{i,20}\, x^2\, t^2 \quad (2)$$

with $x_1{}^2 = y^{p_1}t^{p_2}$ and $x_2{}^2 = x^{p_1}z^{p_2}$. But this system can also be viewed as a multi-homogeneous of type (2; 2), then all the monomials are present.

Example 3:

Consider an unmixed generic polynomial system

$$c_{i1} + c_{i2}\, y^2 + c_{i3}\, z + c_{i4}x^3 + c_{i5}x^3y^2 + c_{i6}x^3z + c_{i7}\, x^2y. \quad (3)$$

Clearly this system is not a multi-graded polynomial system, it is the direct sum of basis supports of 1, y^2,z and exponent vectors of 1, x^3.

Definition 2 [2]:

The cyclic n-roots for every n $\in$ N (n $\geq$ 2) as the solutions X= (X_0, . . . , X_{n-1}) $\in \mathbb{C}^n$ to the following n polynomial equations:

$$X_0+X_1 + \dots + X_{n-1} = 0$$

$$X_1+X_1\, X_2+\dots\dots +X_{n-1}\, X_0 = 0$$

$$\vdots$$

$$X_0X_1 \dots . X_{n-2} + \cdots + X_{n-1}X_0 \dots . X_{n-3} = 0$$

$$X_1 \dots . X_{n-1} = 1 \quad (4)$$

This system of equations is invariant under cyclic permutation of the indices (0,1,...,n−1).

Here we are interested just in bounding the number of isolated solutions.

Example 4:

The cyclic 5-roots problem takes the following form:

$$f(z)= \begin{cases} z_1+z_2+z_3+z_4+z_5=0 \\ z_1z_2+z_2z_3+z_3z_4+z_4z_5+z_5z_1=0 \\ x_2x_3+x_2x_3x_4+x_3x_4x_5+x_4x_5x_1+x_5x_1x_2=0 \\ z_1z_2z_3z_4+z_2z_3z_4z_5+z_3z_4z_5z_1+z_4z_5z_1z_2+z_5z_1z_2z_3=0 \\ z_2z_3z_4z_5-1=0 \end{cases} \quad (5)$$

Observe the symmetry in the system.

For $2 \leq n \leq 9$,the total number γ(n) of cyclic n-roots are given by the table:

n	2	3	4	5	6	7	8	9
γ(n)	2	6	∞	70	156	924	∞	∞

Note : Ralf Froberg conjectured that $\gamma(q) = \binom{2q-1}{q-1}$ for all prime numbers.

3.THE STRUCTURE OF NEWTON POLYTOPE

Definition 3 [2]:

A set K in $\mathbb{R}^n$is said to be convex if it contains the line segment connecting any two points in K.

Definition 4 [2]:

The convex hull of the a support B_i= supp(f)={$\alpha_1,\dots,\alpha_n$}is defined as

Conv(B_i)={$\sum_{i=1}^{n} \lambda_i \alpha_i$: $\sum_{i=1}^{n} \lambda_i = 1$, $\lambda_i \geq 0$ for i=1, ..., n}. (6)

If the set is not convex then the convex hull is the smallest convex set containing it.

We say that S $\subseteq \mathbb{R}^n$ is a convex combination of $s_1,\dots,s_n \in \mathbb{R}^n$ if there exist λ_1 ,….., λ_n such that :

(i) S=$\lambda_1 s_1$+.....+$\lambda_n s_n$,
(ii) $\lambda_1 +$ ....+$\lambda_n = 1$,
(iii) $\lambda_1 \geq 0$,........, $\lambda_n \geq 0$.

Example 5:

Let B={(0,0),(2,0),(0,5),(1,1)} $\subset \mathbb{R}^2$. The triangle with vertices (2,0),(0,0) and(0,5) is Conv(B)

Note that (1,1) = $\frac{3}{10}$ (0,0) + $\frac{1}{12}$ (2,0) + $\frac{1}{15}$ (0,5) is a convex combination of the other three points in B.

Definition 5 [1]:

A polytopes is convex hull of a finite set in $\mathbb{R}^n$. If the finite set is B={$m_1, \dots, m_l$ } as subset of $\mathbb{R}^n$, then the corresponding polytope can be expressed as

Conv(B)={ $\lambda_1 m_1 + \dots + \lambda_l m_l$: $\lambda_i \geq 0, \sum_{i=1}^{l} \lambda_i = 1$}.

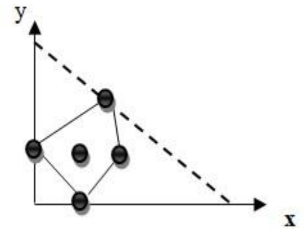
The most important polytopes are convex hulls of sets of points with integer coordinates and these are called lattic polytopes. So lattic polytope is a set of the form Conv (B), where B $\subset$ □ ^{n} is finite.

Definition 6 [1]:

The Newton polytope of f_i is the convex hull of support $A_i, denoted$ by

$Q_i =$ Conv(B_i) $\subseteq \mathbb{R}^n$ or N P(f_i) = Conv(B_i) $\subseteq \mathbb{R}^n$.

The Newton polytope records the “shape” or “sparsity structure” of a polynomial and it tell us which monomials appear with nonzero coefficients.



Figure(2) The Newton polytope of polynomial a_1y + a_2 x^2y^2 + a_3x^2y + a_4x + a_5xy.

Suppose there is a finite set of exponents B={ α_1,…….., α_l} ⊂ □$^{n}_{\geq 0}$ and let L(B) be the set of all polynomials whose terms all have exponents in A,then,

L (B) = {c_1 x^{α_1}+……….+$c_l x^{\alpha_l}$: $c_i \in$ □ } is a vector space over □ of dimension L (the number of elements inB).

For example if B = { (0,0) , (0,1) , (0,2) , (1,0) , (1,1) , (2,0)} then

$$L\ (B) = \{ c_1 + c_2y + c_3y^2 + c_4x + c_5\ xy + c_6x^2\}. \qquad (7)$$

<u>Definition</u> 7 as in [3]:

The Minkowski sum C+D of sets C and D in $\mathbb{R}^n$is C+D={a+b: aϵC, bϵD} $\subset \mathbb{R}^n$.

Equivalently,

$$C+B= U_{\alpha\in C}(C+D). \qquad (8)$$

If Cand D are convex polytopes then C+D is a convex polytope. Note that, the Minkowski sum of convex polytopes can be computed as convex hull of all sums (a+b) of vertices of Cand D respectively.

Definition 8 as in [3]:

If C, B be sets in $\mathbb{R}^n$ and $\lambda \in \mathbb{R}, then$

(I) $The\ scalar\ multiple\ of\ a\ set\ C \subseteq$ $\mathbb{R}^n$ by a real number $\lambda \in \mathbb{R}$ is $\lambda C = \{\lambda a : a \in C\} \subset \mathbb{R}^n$.

(II) The Minkowski difference for the sets C and B is C-B={x ∈ $\mathbb{R}^n$:x+B$\subset C\} \subset \mathbb{R}^n$.

If C and B are convex polytopes then C-B is also a convex polytope.Note thatC– B does not equal C+ (- B) because the first lies in the interior of Cand,

in general B + (C– B) $\subset_{\neq} C$. When C itself is a Minkowski sum B + A, then (B +A) – B =A, for any convex polytope A.

Example 6:

Given the following polynomials

f(z,y)=az^3y^2+bz +cy^2+d , g(z,y)=ezy^4+lz^3+gy (9)

The Minkowski sum of the Newton polytopes

$A_1 = NP(f)$ =Conv{(3,2),(1,0),(0,2),(0,0)} and

$A_2 = Np(g) = Conv${(1,4),(3,0),(0,1)} is a convex heptagon with these vertices (0,1),(3,0),(4,0),(6,2),(1,6) and (0,3)

where

$A_1 + A_2 =$ {(4,6),(2,4),(1,6),(1,4),(6,2),(4,0),(3,2),(3,0),(3,3),(1,1),(0,3),(0,1)}

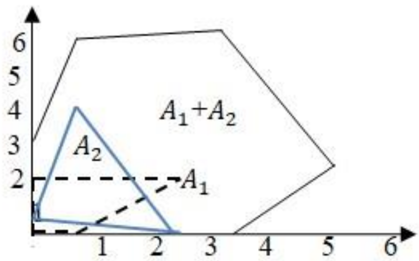


Figure 3 :The Minkowski sum of polytopes

<u>Example</u> 7:

Let any polynomial of the form

f= cxy+dx^2+hy^2+k (10)

, with c,d,h and k≠0 has Newton polytope equal to the triangle

N P$(f) = C$onv$(\{(1,1),(2,0),(0,5),(0,0)\})$.

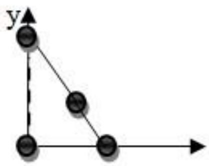


Figure 4 : The supports and the Newton polytops.

<u>Example</u> 8 as in [1]:

This is a system of 3 polymomials in 2 unknowns

$$f_1 = a_{11} + a_{12}xy + a_{13}x^2y + a_{14}x$$

$$f_2 = a_{21}y + a_{22}x^2y^2 + a_{23}x^2y + a_{24}x$$

$$f_3 = a_{31} + a_{32}y + a_{33}xy + a_{34}x \quad (11)$$

Here the Newton polytops shown in the following figure.

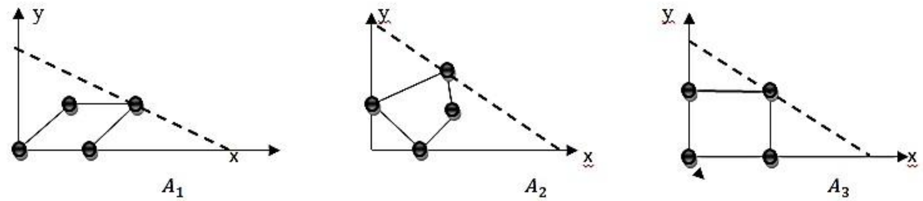


Figure 5 : The supports and the Newton polytops.

Note that the polynomial

f= $c_{21}y+c_{22}x^2y^2+c_{23}x^2y+c_{24}x+c_{25}xy$ (12)

has different support but it has the same Newton polytope as f_2,where A_1=supp(f_2)={(0,1),(2,2),(2,1),(1,0)}
and A_2=supp(f)={(0,1),(2,2),(2,1),(1,0),(1,1)} , but conv(f_2)= A_2 as conv(f).

<u>Definition</u> (9) as in [4]:

The Given convex polytopes A_1,…., $A_n \subseteq \mathbb{R}^n$, there is a unique, up to multiplication by scalar, real-valued function MV(A_1,…., A_n), called the mixed volume of A_1,…., A_n which is multiplication with respect to Minkowski addition and scalar multiplication.In other words, for $\mu, \rho \in \mathbb{R}_{\geq 0}$ and convex polytope $A'_k \subseteq \mathbb{R}^n$

MV $(A_1, \dots, \mu A_k + \rho A'_k, \dots, A_n)$ = μ MV $(A_1, \dots, A_k, \dots, A_n)$ $+\rho\, MV\,(A_1, \dots, A'_k, \dots, A_n)$. (13)

Exactly to define mixed volume we require that

MV (A_1,…., A_n)=n! Vol (A_1), (14)

when A_1=……=A_n. It is clear that the mixed volume of polytopes with integer vertices is integer. An equivalent definition is

Definition (10) as in [4]:

For $c_1,\ldots.. c_n \in \mathbb{R}_{\geq 0}$ and convex polytopes $B_1,\ldots.,$ $B_n \subseteq \mathbb{R}^n$,the mixed volume MV($B_1,\ldots.,$ B_n) is the coefficient of $c_1 c_2 \ldots.. c_n$ in Vol($c_1 B_1+\ldots+c_n B_n$) expanded as a polynomial in $c_1,\ldots.. c_n$. An explicit expression for the mixed volume is obtained by the Exclusion-I nclusion principle:

MV ($B_1,\ldots., B_n$)=$\sum_{I C\{1,\ldots,N\}} (-1)^{n-|I|} Vol(\sum_{i\in I} B_i)$, (15)

Where the second sum is Minkowski addition and |I| denoted set cardinality.

In the case of two polytopes in $\mathbb{R}^2$, the last formula reduces to:

MV (O_1+O_2)=Vol(O_1+O_2)-Vol(O_1)- Vol(O_2) as in [2]. (16)

Where "Vol" is referred to usual area measure of plane Euclidean geometry.

Example (9):

The mixed volumes of polynomial systems are as follows:

MV(O_1, O_2)= Vol(O_1+O_2)-Vol(O_1)- Vol(O_2)=4,

MV(O_1, O_3)= Vol(O_1+O_3)-Vol(O_1)- Vol(O_3)=3

MV(O_2, O_3)= Vol(O_2+O_3)-Vol(O_2)- Vol(O_3)=4

(i) MV(O_1, O_2)=4 (ii) MV(O_1, O_3) (iii) MV(O_2, O_3)=4

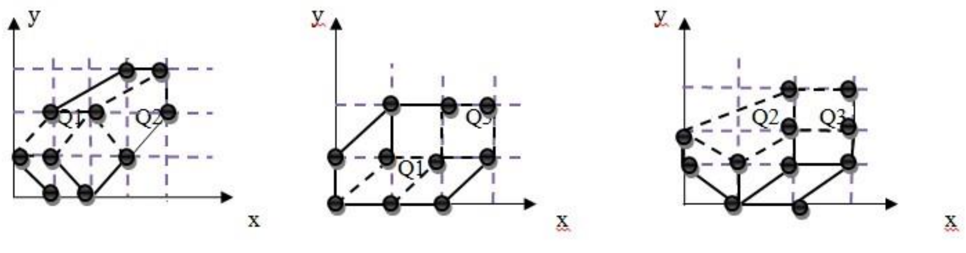


Figure 6 This is the Mixed volumes of convex-hull.

Definition (11) as in [5]

The n-dimensional mixed volume of a collection of polytoes $q_1,\ldots, q_n$, denoted

$MV_n(q_1,\ldots, q_n)$

Is the coefficient of the monomial $\lambda_1\ldots\lambda_2\ldots\lambda_n$ in $Vol_n(\lambda_1 q_1+\ldots+\lambda_n q_n)$ [6].

Example (10):

Let A_1, A_2 are polytopes in $\mathbb{R}^2$ where Q_1 is the unit square conv({(0,0) , (1,0) , (0,1) , (1,1)}) and Q_2 is the triangle Conv({ (0,0) , (1,0) , (0,1)}),

Then $MV_2(A_1, A_2)=2$, because

$$Vol_2(\lambda_1 A_1+\lambda_2 A_2) = \lambda_1(\lambda_1) + \lambda_1\lambda_2 + \lambda_1\lambda_2 + \frac{1}{2}(\lambda_2)(\lambda_2) = \lambda_1^2 + 2\lambda_1\lambda_2 + \frac{1}{2}\lambda_2^2. \quad (17)$$

So, from the definition of mixed volume, the mixed volume of polytopes A_1, A_2 is two as the coefficient of the monomial $\lambda_1\lambda_2$ in $Vol_2(\lambda_1 A_1+\lambda_2 A_2)$.

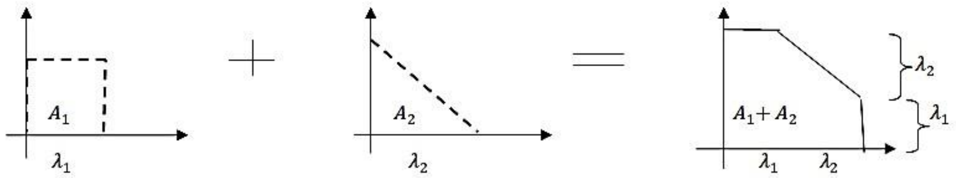


Figure 7: This is the Mixed volume

Newton polytopes provide a model for the sparsity of polynomial systems based on Bernstein's upper bound on the number of common roots . This barrier is also known as the BKK barrier due to the work in [Bern75], [Kush 76], and [Khov78].

<u>Theorem</u> (1) [Bernstein75] [6]:

Let f_1,….. $f_n \in$ □ $[x_1, {x_1}^{-1} \ldots, x_n, {x_n}^{-1}]$ with Newton polytopes Q_1,…., Q_n $\subseteq \mathbb{R}^n$ the number of common solutions in (□ $^*)^n$ is either infinite, or does not exceed MV(Q_1,…., Q_n).Equality holds when all coefficients are generic,for almost all specializations of the coefficients c_{ij} the number of solutions is exactly MV(Q_1,…., Q_n).

The mixed volume is at the heart of sparse elimination as it models the concept of sparsity and provides a limit on the number of isolated roots.

Let A_i=supp (f_i)={ $a_{i1}, \ldots, a_{i\mu_i}$ $\subseteq$ □ ^{n} denote the set of exponent vectors corresponding to monomial in f_i with nonzero coefficients.This set is the support of f_i and

$$f_i=\sum_{j=1}^{\mu_i} c_{ij}\, x^{a_{ij}} , \quad c_{ij} \neq 0, \text{j=1,……, } \mu_i, \qquad (18)$$

. So A_i is uniquely defined given f_i.

<u>Theorem</u> (2) [Bezout's Theorem][7]:

If the homogenous equations f_1=…..= f_{n-1}=0 have degrees d_0,….. d_{n-1} and finitely many solutions in projective space, $\mathbb{P}^n$, then the number of solutions (counted with multiplicity) is d_0….. d_{n-1}.

<u>Example</u> (11) :

Let: f_1(x,y) = ax^{3}y^2 + bx + cy^2 + d = 0,

$$f_2(x,y) = e\, xy^4 + fx^3 + gy=0 , \qquad (19)$$

where a,b,c,d,e,f and g are coefficients in □ , since the polynomials f_1 and f_2 have total degree five, Bezout's Theorem predict that the above system should have 5.5 = 25 solutions in $\mathbb{P}^2$. So, it is important to realizes that by

Bezout’s Theorem, generic equations f_1= f_2= 0 of total degree 5 in x,y have 25 solutions in 2.

every Newton polytope is a scaled copy of the n-dimensional unit simplex with the vertex set, then these two bounds (Bezout bound and BKK bound) are equal

{ (0,…..,0) , (1,0,….,0) , …. , (0,….,0,1) }.

Definition (12) [1]:

“The sparse resultant R=R (A_1,…., A_{n+1}) of system (*) is a polynomial in [c]. If codim (Z)= 1 ,then R (A_1,…., A_{n+1}) is the defining irreducible polynomial of hyper surface Z. If codim(Z) ≻ 1 then R (A_1,…., A_{n+1}) $= 1$.

Let deg_{f_i}R denote the degree of the resultant R in the coefficients of polynomial f_i and let

MV_{-i} = MV (Q_1,…,Q_{i-1},Q_{i+1},…,Q_{n+1}) for i ∈ {1,……, n+1}” [1].

As consequence of Bernstein’s theorem is

Theorem [PS93] (3) [2]:

“The sparse resultant is separately homogeneous in the coefficients c_i of each f_i and its degree in these coefficient equals the mixed volume of the other n Newton polytopes, denoted MV_{-i},

i.e. deg_{f_i}R= MV (Q_1,…,Q_{i-1},Q_{i+1},…,Q_{n+1})= MV_{-i} ,i=1,2,…..,n+1. (20)

The total degree resultant is deg=$\sum_{i=1}^{n+1} MV_{-i}$," (21)

f all Newton polytopes are n-simplices, then the sparse composition results are identical to the homogeneous composition results, scaled by the total degree of the corresponding polynomials, and the polynomials are homogenized.

Example (12):

Consider an unmixed generic polynomial system : a_{i1}+ a_{i2} y^2+ a_{i3} z + $a_{i4}x^3$+ $a_{i5}x^3y^2$ + $a_{i6}x^3$z + a_{i7} x^2y.

the sparse resultant has total degree 4+3+4=11, whereas the homogeneous resultant has total degree12+6+8= 26.The latter can be obtained as the sparse resultant when the Newton polytopes are the triangles in figure 3.

Definition (13) [7]:

"Given a partition of variables (l_1,…,l_r) , let

$(m_1,\ldots\ldots, \; m_r) = \{ (p_{1,1},\ldots.., p_{1,l1}, \; p_{2,1},\ldots.., p_{2,l2},\ldots.. p_{r,1},\ldots, p_{r,lr}) : \sum_{j=1}^{Li} p_{i,j} \leq m_i$,i=1,….,r}.

The r-tuple (m_1,……, m_r) is called a multi-index" [7].

Fix positive integers l_1,…,l_r and set l=l_1+…+l_r suppose we are given L+1 generic polynomials $f_0, f_1 \ldots, f_l \in$ S(d_1,…,d_r) a subspace of polynomials for every r-tuple of nonnegative integers m_1,…,m_r we consider the linear map

$\emptyset$:S$(m_1, \ldots, m_r)^{l+1} \rightarrow$ S(d_1+m_1,……, d_r+m_r),

$(g_0,\ldots.., g_l) \rightarrow f_0g_0 + f_1g_1 + \cdots + f_lg_l$

If the two vector spaces have the same dimension, then $\emptyset$ is represented by a square matrix and R($f_0, f_1 \ldots, f_l$) divides det($\emptyset$).We say that a multiindex $(m_1, \ldots, m_r)$ has the Sylvester property if the two above spaces have the same diminesion and R ($f_0, f_1 \ldots, f_l$) = det($\emptyset$). (22)

Theorem (4) [3]:

Suppose that l_k=1 or d_k=1 for k=1,2,….,r Then the multigraded resultant of type (l_1,… , l_r; d_1, … , d_r) has at least r! different Sylvester type formulas.

For any permutation π of {1,2,…..,r} we construct the corresponding multiindex (m_1,….., m_r) by using the following rule:

$$m_k=(d_k - 1)l_k+d_k \sum_{j:\pi(j) \prec \pi(k)} l_j. \qquad (23)$$

<u>Theorem</u> (5) [3]:

Suppose that l_k=1 or d_k=1 for k=1,2,….,r Then for every permutation π of {1,2,…..,r} the multiindex (m_1,….., m_r) defined by a bove rule satisfies the Sylvester property with respect to $(l_1,\dots,l_r; d_1,\dots,d_r)$.

This is special case for Sylvester resultant of two binary forms of degree d corresponding to the special case r=1, l_1 =1, d_1=d, m=d-1.

<u>Example</u>(13):

Consider the generic polynomial system as follow:

For i = 0, 1, 2 $c_{i,00} + c_{i,01}y + c_{i,02}\, y^2 + c_{i,10}\, x + c_{i,11}\, xy + c_{i,20}\, x^2$.

This system is multigraded of type (1,1;2,2). By computing sparse matrix (greedy version) by using maple multires package there are three of sparse matrices of size 14, 15 and 21. Note that r = 2 in this example so there are at least 2! of different Sylvester type formulas.

4. CONCLUSION

We present in this paper the definitions of multihomogeneous and multigraded polynomial system with some examples of specific type of multigraded bivariate polynomial systems. Every multigraded system is a special case of the multihomogeneous system such that if a multihomogeneous system is of type $(l_1, l_2, \dots, l_r, h_1, h_2, \dots. h_r)$ $, then\ it\ is\ a$ Multigraded system if for each i=1,…..,r either $l_i = 1\ \ or\ h_i = 1$.

Also in this paper we analyzed the mixed volume of a polynomial system in terms of the mixed cells of a mixed subdivision. In addition, the relationship between Newton polytopes of a polynomial system and Minkowski sum of the given Newton polytopes is also described.

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