PRECISE TRAVELING WAVE SOLUTIONS FOR NONLINEAR SCHRÖDINGER EQUATIONS

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ABSTRACT

This paper addresses challenges associated with the study of nonlinear partial differential equations. Various methods have been utilized for the analytical resolution of these equations. In this discussion, we implement the auxiliary equation method, specifically using the auxiliary equation $(\varphi'(\xi))^2 = a\varphi^2(\xi)+b\varphi^4(\xi)+c\varphi^6(\xi)$, to derive the exact traveling wave solutions for two advanced nonlinear Schrödinger equations.

Keywords: Auxiliary equation, traveling wave, Schrödinger equations.

1. INTRODUCTION

Partial differential equations (PDEs) are foundational in describing a myriad of phenomena across various fields such Templates as physics and engineering. These equations provide a mathematical framework for understanding dynamics such as heat flow in physics; wave propagation in optics; and population modeling in ecology, where PDEs have notably governed theoretical developments [1,2]. Since the second half of the 19th century, the investigation into PDEs has attracted substantial attention from mathematicians. Although the study of nonlinear PDEs dates back centuries, there have been significant advancements in the latter half of the 20th century, particularly driven by the need to understand nonlinear wave propagation phenomena [3]. Traveling wave solutions—permanent form solutions that travel at a constant velocity—are particularly crucial in the sciences and engineering for resolving nonlinear PDEs.

Over the past several decades, numerous effective techniques have been developed for deriving precise solutions, including methods like the inverse scattering transform, Hirota's method [4], truncated Painlevé expansion [5], Bäcklund transformation, the exp-function method [6], the simplest equation method [7], the Weierstrass elliptic function method [8-9], and the Jacobi elliptic function method. This paper aims to leverage the auxiliary equation method and its extended form to accurately solve for traveling wave solutions in two complex nonlinear Schrödinger equations.

2.THE ADVANCED AUXILIARYBEQUATION TECHNIQUE

For a specified nonlinear partial differential equation (NLPDE) with independent variables. (x, t) and dependent variable u:

$$H(u, u_x, u_t, u_{xx}, u_{tt}, \ldots) = 0, \qquad (1)$$

Where H is a polynomial in u and in its partial derivatives. Using the traveling wave transformation

$$u(x,t) = u(\xi) \qquad \xi = k(x - \omega t), \qquad (2)$$

where k and ω are constants, to reduce Eq. (1) to the following nonlinear ODE:

$$G(u, u', u'', \ldots) = 0, \qquad (3)$$

Where G is a polynomial in $u(\xi)$ and its total derivatives, where $' = \frac{d}{d\xi}$.

We assume that Eq. (3) has the formal solution

$$u(\xi) = F(\phi(\xi)), \qquad (4)$$

where F is a suitable variable transformation, and $\phi(\xi)$ is the solution of the first order ODE:

$$(\phi'(\xi))^2 = c_0 + c_2\phi^2(\xi) + c_4\phi^4(\xi) + c_6\phi^6(\xi), \quad (5)$$

Where c_i $(i = 0, 2, 4, 6)$ are arbitrary constants to be determined. Equation (5) has the solutions

$$\phi(\xi) = \frac{1}{2}\left[-\frac{c_4}{c_6}(1 \pm f(\xi)\right]^{\frac{1}{2}}, \quad (6)$$

The functions $f(\xi)$ given by (6) have twelve forms as follows:

(1) If $c_0 = \frac{c_4^3(m^2-1)}{32c_6^2m^2}$, $c_2 = \frac{c_4^2(m^2-1)}{16c_6m^2}$, $c_6 > 0$, then

$$f(\xi) = sn(\rho\xi). \text{ Or } f(\xi) = \frac{1}{m\, sn\,(\rho\xi)}, \quad (7)$$

where $\rho = \frac{c_4}{2m}\sqrt{\frac{1}{c_6}}$.

(2) If $c_0 = \frac{c_4^3(1-m^2)}{32c_6^2}$, $c_2 = \frac{c_4^2(5-m^2)}{16c_6}$, $c_6 > 0$, then

$$f(\xi) = m\, sn(\rho\xi). \text{ Or } f(\xi) = \frac{1}{sn\,(\rho\xi)}, \quad (8)$$

where $\rho = \frac{c_4}{2}\sqrt{\frac{1}{c_6}}$.

(3) If $c_0 = \frac{c_4^3}{32m^2c_6^2}$, $c_2 = \frac{c_4^2(4m^2+1)}{16c_6m^2}$, $c_6 < 0$, then

$$f(\xi) = cn(\rho\xi). \text{ Or } f(\xi) = \frac{\sqrt{1-m^2}sn(\rho\xi)}{dn\,(\rho\xi)}, \quad (9)$$

where $\rho = \frac{-c_4}{2m}\sqrt{\frac{-1}{c_6}}$.

(4) If $c_0 = \frac{c_4^3 m^2}{32c_6^2(m^2-1)}$, $c_2 = \frac{c_4^2(5m^2-4)}{16c_6(m^2-1)}$, $c_6 < 0$, then

$$f(\xi) = \frac{dn(\rho\xi)}{\sqrt{1-m^2}} \quad \text{or} \quad f(\xi) = \frac{1}{dn\,(\rho\xi)}, \tag{10}$$

where $\rho = \frac{c_4}{2}\sqrt{\frac{1}{c_6(m^2-1)}}$.

(5) If $c_0 = \frac{c_4^3}{32c_6^2(1-m^2)}$, $c_2 = \frac{c_4^2(4m^2-5)}{16c_6(m^2-1)}$, $c_6 > 0$, then

$$f(\xi) = \frac{1}{cn\,(\rho\xi)}, \quad \text{or } f\,(\xi) = \frac{dn(\rho\xi)}{\sqrt{1-m^2}sn(\rho\xi)} \tag{11}$$

where $\rho = \frac{c_4}{2}\sqrt{\frac{1}{c_6(1-m^2)}}$.

(6) If $c_0 = \frac{m^2 c_4^3}{32c_6^2}$, $c_2 = \frac{c_4^2(m^2+4-)}{16c_6}$, $c_6 < 0$, then

$$f(\xi) = dn(\rho\xi). \quad \text{Or } f(\xi) = \frac{\sqrt{1-m^2}}{dn(\rho\xi)}, \tag{12}$$

where $\rho = -\frac{c_4}{2}\sqrt{\frac{-1}{c_6}}$.

3.THE ADVANCED NONLINEAR SCHRODINGER EQUATION

We seek solutions for the equation presented below

$$q_z = i\alpha_1 q_{tt} + i\alpha_2|q|^2 + \alpha_3 q_{ttt} + \alpha_4(q|q|^2)_t + \alpha_5 q(|q|^2)_t, \tag{13}$$

Eq. (13) can be rephrased as:

$$q(z,t) = u(\xi)e^{[i(kz+wt)]}, \quad \xi = t + C_z, \tag{14}$$

where $u(\xi)$ is a function of ξ while k, ω *and* C are nonzero constants.

By substituting Equation (14) into Equation (13) we have the following result:

Im : $(\alpha_1 + 3\omega\alpha_3)u'' - (\alpha_3\omega^3 + \alpha_1\omega^2 + k)u + (\alpha_2 + \alpha_4\omega)u^3 = 0,$ (15)

Re : $\alpha_3 u''' - (2\alpha_1\omega + 3\alpha_3\omega^2 + C)u' + (3\alpha_4 + 2\alpha_5)u^2u' = 0,$ (16)

Integrating Eq. (16) we obtain

$$\alpha_3 u'' - (2\alpha_1\omega + 3\alpha_3\omega^2 + C)u + \frac{1}{3}(3\alpha_4 + 2\alpha_5)u^3 = 0. \qquad (17)$$

The necessary and sufficient condition for both Eqs. (14) and (17) is the relation as follows:

$$\omega = \frac{3\alpha_2\alpha_3 - \alpha_1(3\alpha_4 + 2\alpha_5)}{6\alpha_3(\alpha_4 + \alpha_5)}, \qquad (18)$$

$$k = 8\alpha_3\omega^3 + 8\alpha_1\omega^2 + \frac{2\alpha_1^2 + 3\alpha_3 C}{\alpha_3}\omega + \frac{\alpha_1}{\alpha_3}. \qquad (19)$$

For simplicity, Eq. (17) can be written as:

$$u''(\xi) + k_1 u(\xi) + k_3 u^3(\xi) = 0, \qquad (20)$$

Where

$$k_1 = -\frac{2\alpha_1\omega + 3\alpha_3\omega^2 + C}{\alpha_3}, \qquad k_3 = \frac{3\alpha_4 + 2\alpha_5}{3\alpha_3}. \qquad (21)$$

By balancing between $u''\ with\ u^3$ in (20) we get

$$u(\xi) = a_0 + a_1\phi(\xi) + a_2\phi^2(\xi), \qquad (22)$$

where $a_0, a_1\ and\ a_2$ are constants that need to be determined.

Substituting Equation (22) into Equation (20) results a system of algebraic equations:

$\phi^6 :\ a_2^3 k_3 + 8ca_2 = 0,$

$\phi^5:\ 3a_1a_2^2k_3 + 3ca_1 = 0,$

$\phi^4:\ 3a_0a_2^2k_3 + 3a_1a_2^2k_3 + 6ba_2 = 0,$

$\phi^3:\ 6a_0a_1a_2^2k_3 + a_1^3k_3 + 2ba_1 = 0,$

$\phi^2:\ 3a_0a_2^2k_3 + 3a_0a_2^2k_3 + 4aa_2 + a_2k_1 = 0,$

$\phi:\ 3a_0a_2^2k_3 + aa_1 + a_1k_1 = 0,$

$\phi^0:\ a_0^2k_3 + a_0k_1 + 6ba_2 = 0,$

By solving the equations with Maple, the following results are obtained:

Result 1:

$$a_0 = \pm\sqrt{-\frac{k_1}{k_2}},\quad a_1 = 0,\ a_2 = \pm\sqrt{-\frac{8c}{k_3}},\quad a = \frac{1}{2}k_1,\ b = \pm\sqrt{2ck_1},\quad (23)$$

From (22) and (23), we deduce the optical Soliton solutions as follows:

$$q_1(\xi) = \pm\sqrt{-\frac{k_1}{k_3}}\left(1 - \frac{sech^2(\xi\sqrt{\frac{1}{2}k_1})}{1-\frac{1}{4}(1\pm tansh(\xi\sqrt{\frac{1}{2}k_1}))^2}\right)e^{[i(kz+\omega t)]},$$

(24)

$$q_2(\xi) = \pm\sqrt{-\frac{k_1}{k_3}}\left(1 + \frac{csch^2(\xi\sqrt{\frac{1}{2}k_1})}{1-\frac{1}{4}(1\pm coth(\xi\sqrt{\frac{1}{2}k_1})^2}\right)e^{[i(kz+\omega t)]},$$

(25)

$$q_3(\xi) = \pm\sqrt{-\frac{k_1}{k_3}}\left(1 - \frac{sech^2(\xi\sqrt{\frac{1}{2}k_1})}{1-\frac{1}{4}(1\pm tansh(\xi\sqrt{\frac{1}{2}k_1}))}\right)e^{[i(kz+\omega t)]},$$

(26)

$$q_4(\xi) = \pm\sqrt{-\frac{k_1}{k_3}}\left(1 + \frac{csch^2(\xi\sqrt{\frac{1}{2}k_1})}{1-\frac{1}{4}(1\pm coth(\xi\sqrt{\frac{1}{2}k_1})}\right)e^{[i(kz+\omega t)]} \quad , \tag{27}$$

$$q_5(\xi) = \pm\sqrt{-\frac{k_1}{k_3}}\ tanh(\xi\sqrt{\frac{1}{2}k_1})e^{[i(kz+\omega t)]} \quad , \tag{28}$$

$$q_6(\xi) = \pm\sqrt{-\frac{k_1}{k_3}}\ coth(\xi\sqrt{\frac{1}{2}k_1})e^{[i(kz+\omega t)]} \quad , \tag{29}$$

$$q_7(\xi) = \pm\sqrt{-\frac{k_1}{k_3}}\ (1 + \frac{16\sqrt{2ck_1}}{exp(\pm\xi\sqrt{2k_1})-8\sqrt{2ck_1}})e^{[i(kz+\omega t)]} \quad , \tag{30}$$

Result 2:

$$a = -\frac{1}{4}k_1,\ \ b = 0,\ a_0 = 0,\ a_1 = 0, a_2 = \pm\sqrt{-\frac{8c}{k_3}} \quad ,$$

(31)

In this result, we deduce the optical Soliton solutions as follows:

$$q_8(\xi) = \pm\left(\frac{8k_1\sqrt{-\frac{2c}{k_3}}exp(\pm\xi\sqrt{-k_1})}{1+16k_1 cexp(\xi 2\sqrt{-k_1})}\right)e^{[i(kz+\omega t)]}, \tag{32}$$

4. CONCLUSION

The auxiliary equation method was employed to derive exact traveling wave solutions for higher order nonlinear Schrodinger equations. The solutions achieved via this method include three distinct: hyperbolic, trigonometric and rational solutions. These solutions were verified using Maple 18 by substituting them back into the original equations.

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