APPLYING THE DOMINANT BALANCE METHOD TO SOLVE VARIABLE COEFFICIENTS 2ND ORDER LINEAR ORDINARY DIFFERENTIAL EQUATION AROUND IRREGULAR SINGULAR POINT

Siham Aldwibi

Mathematics Department, Faculty of Education, Alasmarya Islamic University
Zliten, Libya
Corresponding email address: siham.aldwibi@asmarya.edu.ly

Published online: 12 May 2024

ABSTRACT

Ordinary differential equations with variable coefficients are, in general, difficult to solve, specifically finding exact solutions to such type of equations. Thus, approximation solutions are excellent alternatives to exact solutions. There are many types of approximation methods that are available to applied mathematicians to apply. Among these approximation methods are asymptotic methods which include Dominant Balance Method (DBM). In this paper, we apply the Dominant Balance Method to solve 2^{nd} order variable coefficients ordinary differential equations when Frobenius method fails, that happens when we have irregular singular point. The method has been discussed in details and applied to equations and the solutions obtained by the asymptotic method are represented by the first few terms of asymptotic sequence. The reader has to be aware of the perturbation theory and its symbols because they are keys to understand the steps of application.

Keywords: *Asymptotic solutions, approximation, dominant balance, irregular singular point, second order differential equations.*

1. INTRODUCTION

Asymptotic methods, as one type of theoretical methods, can provide results that lead to obtaining effective numerical evaluation. Dominant Balance Method (DBM) is one type of asymptotic methods and was explained and used in many papers and books on mathematical methods [1-3].

A dominant balance is a subset of governing equation terms which locally and statistically dominates the remaining equation terms by at least an order of magnitude, so it is balancing as many terms as possible at leading order

[3]. It is used to determine the asymptotic behavior of solutions to an equation without fully solving it [4], and here you can find the exact meaning of asymptotic [1]. The method is not new, it goes back to Newton. This method (DBM) is applicable to any type of equations such as polynomial algebraic equations and ordinary and partial differential equations [3,4]. And it has many applications such as those mentioned in these papers [5-8].

Our interest in this paper is in differential equations with variable coefficients. You apply the (DBM) method once; make some informed guesses about each term: which term is negligible and which term is a controlling factor. Drop the negligible terms and solve the resulting simpler ordinary differential equation. Check that the solution is consistent with the guesses you made. If this is the case, then you have the controlling factor. Otherwise, you need to drop different terms instead. Repeat the process to higher orders. When repeating the process, you will use the first output you obtained as an input in the next step. And the process should be repeated as many as you need to obtain the desirable number of terms in the asymptotic expansion (i.e. the asymptotic solution). Full explanation of the procedure can be found here [3].

2. PRELIMINARIES

For the equation of the form

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0 \qquad (1)$$

Which is variable coefficients 2^{nd} order differential equation; Frobenius Method can be applied around the point ($x = x_0$) to find its solution. If both

$p(x_0), q(x_0)$ were finite, then we call $x = x_0$ ordinary point of equation (1) and we can find two independent particular solutions of equation (1) as power series of the form $y \sim \sum_{n=0}^{\infty} a_n (x - x_0)^n$ as $x \rightarrow x_0$ [6],[9]. If either $\lim_{x \to x_0} p(x) = \infty$ or $\lim_{x \to x_0} q(x) = \infty$ or both of them, then we have a singular point ($x = x_0$ singular point of equation (1)). Here we encounter one of two cases:

Case 1: $\lim_{x \to x_0} (x - x_0) p(x)$ and $\lim_{x \to x_0} (x - x_0)^2 q(x)$ are both finite, then $x = x_0$ is a regular singular point. Then, at least one particular solution of equation (1) can be found as a power series of the form $y \sim (x - x_0)^{\lambda} \sum_{n=0}^{\infty} a_n (x - x_0)^n$ as $x \rightarrow x_0$, λ is constant.

Case 2: if either $\lim_{x \to x_0} (x - x_0) p(x) = \infty$ or $\lim_{x \to x_0} (x - x_0)^2 q(x) = \infty$ or both of them, then we have irregular singular point and Frobenius Method fails [2,9,12]. This is the case of concern to us in this paper.

If the point at infinity is an irregular singular point, the solution must be represented by an asymptotic expansion. The expansions in these cases are obtained using the Dominant Balance Method (DBM). Reference [10-12] discussed this problem and applied asymptotic solution of the form: $y \sim e^{\Lambda(x)} x^{\sigma} u(x) \rightarrow (*)$, where $u(x)$ can be expressed as a power series in $x^{\frac{m}{n}}$,which need not be convergent, and $\Lambda(x)$ is a polynomial in $x^{\frac{-m}{n}}$. Letting λx^{ν} be the leading term in $\Lambda(x)$, substituted the above solution in equation (1) and used the Dominant Balance Method to extract the dominant part of each term, found the value of ν. If ν is an integer then solution $(*)$ is called a normal solution and has the form: $y(x) = \exp(\lambda_{\nu} x^{\nu} + \lambda_{\nu-1} x^{\nu-1} + \ldots + \lambda_1 x)\, x^{\sigma} (1 + a_1 x^{-1} + \ldots)$. If ν is not an integer, solution $(*)$

is called a subnormal solution and Λ is a polynomial in $x^{\frac{1}{2}}$ while $u(x)$ is an ascending series in $x^{-\frac{1}{2}}$ [10-12].

As we notice in the form of solution $(*)$ there are 3 unknowns (λ, ν, σ) have to be found first then use them to find the coefficients of the series $u(x)$ using the recurrence relation obtained from substituting $(*)$ into the given ODE. These are clearly long calculations. In this paper we will replace the form of solution $(*)$ by the form $y(x) \sim \exp\left(\sum_{n=0}^{\infty} \varphi_n(x)\right) \rightarrow (**)$ where $\{\varphi_n(x)\}_{n=0}^{\infty}$ is an asymptotic sequence as $x \rightarrow +\infty$. This clearly requires fewer calculations than the form $(*)$. But first we need to remove the term containing the first derivative in equation (1) in order to use the Dominant Balance Method.

3. MATHEMATICAL FORMULATION

We have the variable coefficients 2nd order differential equation

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0 \qquad (1)$$

And to apply the DBM, this equation has to be written in the form

$$\frac{d^2Y}{dx^2} + q(x)Y = 0 \qquad (2)$$

Any equation of the form of equation (1) can be reduced to equation of the form (2) via the change of dependent variable:

$$Y = y\, exp\left(\frac{1}{2}\int p(x)\,dx\right) \qquad (3)$$

Indeed,

$$Y = y\, exp\left(\frac{1}{2}\int p(x)\, dx\right) \quad \rightarrow \quad y = Y \exp\left(-\frac{1}{2}\int p(x)\, dx\right)$$

$$\rightarrow \frac{dy}{dx} = \left(\frac{dY}{dx} - \frac{1}{2}p(x)Y\right)\exp\left(-\frac{1}{2}\int p(x)\right)dx$$

$$\rightarrow \frac{d^2y}{dx^2} = \left(\frac{d^2Y}{dx^2} - p\frac{dY}{dx} - \frac{1}{2}\frac{dp}{dx}Y + \frac{1}{4}p^2Y\right)\exp\left(-\frac{1}{2}\int p(x)\, dx\right),$$

Substitute the first and second derivative in equation (1), we obtain

$$\left(\frac{d^2Y}{dx^2} - p\frac{dY}{dx} - \frac{1}{2}\frac{dp}{dx}Y + \frac{1}{4}p^2Y + p\frac{dY}{dx} - \frac{1}{2}\, p^2Y + qY\right)\exp\left(-\frac{1}{2}\int p(x)\, dx\right) = 0,$$

$$\rightarrow \frac{d^2Y}{dx^2} + \left(-\frac{1}{2}\frac{dp}{dx} - \frac{1}{4}\, p^2 + q\right)Y = 0,$$

$$\rightarrow \frac{d^2Y}{dx^2} + \tilde{q}(x)Y = 0, \qquad (4),$$

This is in the form of equation (2). Indeed, any equation of the form (1) can be written in the form (4) via the change of variable (3).

Example 1: $\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + \frac{1}{x}\, y = 0$, can be transformed into: $\frac{d^2Y}{dx^2} + \left(-1 - x^2 + \frac{1}{x}\right)Y = 0$, using (3).

Example 2: $\frac{d^2y}{dx^2} - \frac{dy}{dx} + \frac{1}{1+x}\, y = 0$, can be transformed into: $\frac{d^2Y}{dx^2} + \frac{3-x}{4(x+1)}Y = 0$, using (3).

Change of variable

In most of the important equations with irregular singularities, these singularities occur at $x = \infty$. The behavior of analytic differential equations at infinity is here defined as the behavior at $s = \frac{1}{x}$ [4].

Introduce the new variable: $s = \frac{1}{x}$, which can be written as $x = \frac{1}{s}$, and differentiate to get: $dx = -\frac{1}{s^2}\, ds$. Then we can write $\frac{dY}{dx} = -s^2 \frac{dY}{ds}$. Then we can find the second derivative of Y as: $\frac{d^2Y}{dx^2} = s^4 \frac{d^2Y}{ds^2} + 2s^3 \frac{dY}{ds}$, and equation (4) reads as:

$$\frac{d^2Y}{ds^2} + \frac{2}{s}\frac{dY}{ds} + \frac{1}{s^4}\ \tilde{q}\left(\frac{1}{s}\right) Y = 0,$$

which has the form of equation (1) with $p(s) = \frac{2}{s}$, $q(s) = \frac{1}{s^4}\ \tilde{q}\left(\frac{1}{s}\right)$. since $p(s) \xrightarrow{s\to 0} \infty$, then point $s = 0$ is a singular point. Hence, $x = \pm\infty$ is a singular point of equation (4). To decide whether it is a regular or irregular singular point, we need to check the limit of $s^2 q(s)$, $\lim_{s\to 0} s^2 q(s) = \lim_{s\to 0} \tilde{q}\left(\frac{1}{s}\right) = \lim_{x\to\infty} x^2 \tilde{q}\,(x)$. Thus, if $\lim_{x\to\infty} x^2 \tilde{q}\,(x)$ is finite then $x = \pm\infty$ is a regular singular point. If not, then it is irregular singular point as we have seen before and Frobenius Method cannot be used to find solutions as $x \to \infty$. Method of Dominant Balance (DBM) can be applied to find such solutions.

4. APPLYING THE METHOD OF DOMINANT BALANCE

We will assume the particular solution of equation (2) can be written as :

$$y(x) \sim \exp\left(\sum_{n=0}^{\infty} \varphi_n(x)\right) \quad \rightarrow (6)$$

Where $\{\varphi_n(x)\}_{n=0}^{\infty}$ is an asymptotic sequence as $x \rightarrow +\infty$. Substitute (6) into (2) which gives

$$\frac{d^2}{dx^2}\left[\exp\left(\sum\nolimits_{n=0}^{\infty} \varphi_n(x)\right)\right] + q(x)\exp\left(\sum_{n=0}^{\infty} \varphi_n(x)\right) = 0\,, \qquad (7)$$

First derivative

$$\frac{d}{dx}[\exp\ (\textstyle\sum_{n=0}^{\infty} \varphi_n(x))] = \sum_{n=0}^{\infty} \varphi_n{}'(x) \exp\ (\sum_{n=0}^{\infty} \varphi_n(x)),$$

Second derivative

$$\frac{d^2}{dx^2}[\exp\ (\textstyle\sum_0^{\infty} \varphi_n(x))] = \left[\sum_0^{\infty} \varphi_n{}''(x) + \left(\sum_0^{\infty} \varphi_n{}'(x)\right)^2\right]\exp(\sum_0^{\infty} \varphi_n),$$

And equation (7) becomes

$$\sum\nolimits_{n=0}^{\infty} \varphi_n{}''(x) + \left(\sum\nolimits_{n=0}^{\infty} \varphi_n{}'(x)\right)^2 + q(x) = 0, \qquad (8)$$

$$\rightarrow\ \varphi''_0 + \varphi''_1 + \varphi''_2 + \ldots. + (\varphi'_0)^2 + (\varphi'_1)^2 + (\varphi'_2)^2 + \ldots\ldots + 2\varphi'_0\varphi'_1 + 2\varphi'_0\varphi'_2 + \ldots + 2\varphi'_1\varphi'_2 + \cdots + q(x) \simeq 0, \qquad (9)$$

The terms of equation (9) are not all of the same order. Some are dominant over others. Since

$$\varphi_1 = \circ\,(\varphi_0) \qquad as \quad x \rightarrow +\infty$$

$$\varphi''_1 = \circ\,(\varphi''_0) \qquad as \quad x \rightarrow +\infty$$

$$(\varphi'_1)^2 = \circ\,(\varphi'_0)^2 \qquad as \quad x \rightarrow +\infty$$

$$\varphi_0\varphi_1 = \circ\,(\varphi'_0)^2 \qquad as \quad x \rightarrow +\infty$$

$$\varphi_0' \, \varphi_2' = \circ \, (\varphi_0' \, \varphi_1') \qquad as \quad x \to +\infty \,, \;\; etc$$

Where $f(x) =\circ \, g(x)$ if $lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$. For full explanation of this notation see [1].

If we ignore all the terms dominated by φ_0'' and $(\varphi_0')^2$ then equation (9) reduces to

$$\varphi_0'' + (\varphi_0')^2 + q(x) \simeq 0, \qquad (10)$$

However, this may not be the dominant balance equation as φ_0'' may dominate $(\varphi_0')^2$ and otherwise.

If φ_0'' dominates $(\varphi_0')^2$, then the dominant balance is $\varphi_0'' + q(x) = 0.$

If $(\varphi_0')^2$ dominates φ_0'' , then the dominant balance is $(\varphi_0')^2 + q(x) = 0$.

If $\varphi\,_0'' = O((\varphi_0')^2)$ (i.e. $lim \frac{\varphi_0''}{(\varphi_0')^2} = c$), then it is $\varphi_0'' + (\varphi_0')^2 + q(x) = 0.$

In order to choose between these three cases, we integrate them one by one and use the solutions to check the dominance conditions. As soon as the dominance condition is satisfied, we confirm the dominant balance equation and accept the solution for φ_0. Note that if $q(x) = q_0(x) + q_1(x) + \cdots + q_k(x)$, where $q_{i+1} =\circ \, (q_i)$ $as \; x \to \infty$ then only q_0 is used in the three cases above.

Next, remove all the terms (which appear in the equation of dominant balance) from equation (9). Then substitute φ_0 with the solution in all remaining terms of equation (9) which include φ_0. The modified equation (9) has the same structure as the original equation (9) but without φ_0. Then we repeat the procedure and find the dominant balance which determines

$\varphi_1$and so on. Continue until the required number of gauge functions is found.

Example 3

First we show that $x = +\infty$ is an irregular singular point of the equation

$$\frac{d^2y}{dx^2} - \frac{1}{x}\, y = 0\,, \qquad (11)$$

$q(x) = -\frac{1}{x}$. That means $x^2 q(x) = -x$, which tend to ∞ as $x \to +\infty$, so we have irregular singular point.

So the solution cannot be found using Frobenius series. Applying the Dominant Balance Method, we put

$y(x) \sim \exp\left(\sum_{n=0}^{\infty} \varphi_n\,(x)\right)$,Where $\varphi_{n+1} = \circ\,(\varphi_n)$ as $x \to +\infty$,
($i.e$ $\varphi_{n+1} = \circ\,(\varphi_n)$ if $lim_{x\to\infty} \frac{\varphi_{n+1}}{(\varphi_n))} = 0$). Substituting this solution into equation (11) we obtain

$$\sum_{n=0}^{\infty} \varphi_n'' + \left(\sum_{n=0}^{\infty} \varphi_n'\right)^2 - \frac{1}{x} = 0\,,$$

Expanding to obtain

$$\varphi_0'' + \varphi_1'' + \varphi_2'' + \cdots + (\varphi_0')^2 + (\varphi_1')^2 + (\varphi_2')^2 + \cdots + 2\varphi_0'\varphi_1' + 2\varphi_0'\varphi_2' + \cdots + 2\varphi_1'\varphi_2' + \cdots - \frac{1}{x} \simeq 0\,, \qquad (12)$$

Guessing the Dominant Balance equation:

$\boldsymbol{\varphi_0}$: $\varphi_0'' + (\varphi_0')^2 - \frac{1}{x} \simeq 0$,

Suppose that $(\varphi_0')^2$ dominates φ_0'' as $x \to +\infty$, then the dominant balance is

$$(\varphi_0')^2 - \frac{1}{x} \simeq 0\,, \quad (13)$$

$\rightarrow\ (\varphi_0')^2 = x^{-1}$. This gives us $\varphi_0' = \pm x^{-\frac{1}{2}} \ \rightarrow\ \ \varphi_0'' = \pm\frac{1}{2}\,x^{-\frac{3}{2}}$. It is clear that $(\varphi_0')^2$ dominates φ_0'' , so we have the correct dominant balance. Solve the dominant balance equation: $\varphi_0' = \pm x^{-\frac{1}{2}}$ and proceed to find $\varphi_0 = \pm 2x^{\frac{1}{2}}$. We do not add integration constant here because this is a particular solution.

Now remove terms of (13) from (12) and obtain the updated version of (12):

$$\varphi_0'' + \varphi_1'' + \varphi_2'' + \cdots + (\varphi_1')^2 + (\varphi_2')^2 + \cdots + 2\varphi_0'\varphi_1' + 2\varphi_0'\varphi_2' + \cdots + 2\varphi_1'\varphi_2' + .. \simeq 0, \quad (15)$$

Guessing the Dominant Balance equation:

$\boldsymbol{\varphi_1}$: $\varphi_0'' + \varphi_1'' + 2\varphi_0'\varphi_1' \simeq 0\,,$

Suppose that φ_0'' dominates φ_1'' as $x \rightarrow +\infty$, so we have the dominant balance equation

$$\varphi_0'' + 2\varphi_0'\varphi_1' = 0\,, \quad (16)$$

We have $\varphi_0 = \pm 2x^{\frac{1}{2}} \ \rightarrow\ \varphi_0' = \pm x^{-\frac{1}{2}} \rightarrow\ \varphi_0'' = \mp\frac{1}{2}\,x^{-\frac{3}{2}}$, and the dominant balance equation is

$$\varphi_0'' + 2\varphi_0'\varphi_1' = 0 \ \ \rightarrow\ \ \mp\frac{1}{2}\,x^{-\frac{3}{2}} \pm 2x^{-\frac{1}{2}}\varphi_1' = 0 \ \rightarrow\ \ \varphi_1' = \frac{1}{4}\,x^{-1}\,.$$

$\varphi_1'' = -\frac{1}{4}\,x^{-2}$, which is subdominant compared to φ_0'' . So we have the correct dominant balance and proceed to find $\varphi_1 = \frac{1}{4}\ln(x) = \ln(x^{\frac{1}{4}})$.

Now remove terms of (16) from (15) and obtain the updated version of (15), which is

$$\varphi_1'' + \varphi_2'' + \cdots + (\varphi_1')^2 + (\varphi_2')^2 + \cdots + 2\varphi_0'\varphi_2' + \cdots + 2\varphi_1'\varphi_2' + .. \simeq 0, \quad (17)$$

Guessing the Dominant Balance equation:

$$\boldsymbol{\varphi_2}: \quad \varphi_1'' + \varphi_2'' + (\varphi_1')^2 + 2\varphi_0'\varphi_2' \simeq 0 , \quad (18)$$

Since $\varphi_1'' \propto x^{-2}, (\varphi_1')^2 \propto x^{-2}$, then both φ_1'' and $(\varphi_1')^2$ must be kept as new deriving terms in the new dominant balance equation.

But $\varphi_1'' + (\varphi_1')^2 = -\frac{1}{4}x^{-2} + \frac{1}{16}x^{-2} = -\frac{3}{16}x^{-2}$, and $2\varphi_0' = \pm 2x^{-\frac{1}{2}}$ and equation (18) becomes

$$\rightarrow \ -\frac{3}{16}x^{-2} + \varphi_2'' \pm 2x^{-\frac{1}{2}}\varphi_2' \simeq 0 \ , \quad (19)$$

Assume that φ_2'' is subdominant compared to $\pm 2x^{-\frac{1}{2}}\varphi_2'$. Then the dominant balance equation is

$$-\frac{3}{16}x^{-2} \pm 2x^{-\frac{1}{2}}\varphi_2' = 0 \quad \rightarrow \quad \varphi_2' = \pm\frac{3}{52}x^{-\frac{3}{2}}$$

We see that $\varphi_2'' \propto x^{-\frac{5}{2}}$, and $x^{-\frac{1}{2}}\varphi_2' \propto x^{-2}$, hence our assumption is correct and we have the correct dominant balance equation. Solve it to obtain $\varphi_2 = \mp\frac{3}{16}x^{-\frac{1}{2}}$.

We have found two linearly independent asymptotic particular solutions

$$y_{\pm} \sim \exp\left[\varphi_0 + \varphi_1 + \varphi_2 + \circ\left(x^{-\frac{1}{2}}\right)\right]$$

$$= \exp[\pm 2x^{\frac{1}{2}} + \ln\left(x^{\frac{1}{4}}\right) \mp \frac{3}{16}x^{-\frac{1}{2}} + \circ\left(x^{-\frac{1}{2}}\right)]$$

$$= x^{\frac{1}{4}} \exp[\pm \left(2x^{\frac{1}{2}} - \frac{3}{16} x^{-\frac{1}{2}}\right) + \circ \left(x^{-\frac{1}{2}}\right)]$$

$$= x^{\frac{1}{4}} \exp\left[\pm \left(2x^{\frac{1}{2}} - \frac{3}{16} x^{-\frac{1}{2}}\right)\right]. \exp[\circ \left(x^{-\frac{1}{2}}\right)]$$

$$= x^{\frac{1}{4}} \exp\left[\pm \left(2x^{\frac{1}{2}} - \frac{3}{16} x^{-\frac{1}{2}}\right)\right] \left(1 + \circ \left(x^{-\frac{1}{2}}\right)\right).$$

And the general solution of equation (11) can be written as asymptotic series with the first two terms in them are

$$y = Ay_+ + By_-$$

$$= x^{\frac{1}{4}} \left[A\, exp \left(2x^{\frac{1}{2}} - \frac{3}{16} x^{-\frac{1}{2}}\right) + B \exp\left(-2x^{\frac{1}{2}} + \frac{3}{16} x^{-\frac{1}{2}}\right)\right] \left(1 + \circ \left(x^{-\frac{1}{2}}\right)\right). \qquad (20)$$

We can apply this methods to many other 2nd order ODEs with variable coefficients when $x = +\infty$ is an irregular singular point.

Example 4

$$\frac{d^2y}{dx^2} + \left(1 + \frac{1}{x^2}\right) y = 0\,, \qquad (21)$$

We see that $q(x) = 1 + \frac{1}{x^2}$, that means $x^2 q(x) = x^2 + 1$, which $\to \infty$ as $x \to \infty$, so we have irregular singular point.

So the solution cannot be found using Frobenius series. Applying the Dominant Balance Method, we write the solution $y(x)$ as the last example $y(x) \sim \exp\left(\sum_{n=0}^{\infty} \varphi_n\,(x)\right)$, where $\varphi_{n+1} = \circ\,(\varphi_n)$ as $x \to +\infty$, which means that $\varphi_{n+1} = \circ\,(\varphi_n)$ if $lim_{x\to\infty} \frac{\varphi_{n+1}}{(\varphi_n))} = 0$. Substituting this solution into the given equation, and following the same procedure as the previous example we obtain the asymptotic general solution of equation (21)

$$y = Ay_{+} + By_{-} ,$$

$$= e^{-\frac{1}{4x^2}}\left(A\sin\left(x-\frac{1}{2x}\right)+B\cos\left(x-\frac{1}{2x}\right)\right)\left(1+o(x^{-2})\right) \quad \text{as} \quad x \to +\infty \,.$$

5. CONCLUSION

In this paper, we applied the dominant balance method to solve ordinary differential equations of the second order with variable coefficients when Frobenius method fails to give solution in the form of power series. The solution obtained using the dominant balance is asymptotic, which means it is represented by the first few terms of asymptotic sequence and you can apply the method as many as you need to obtain the desired numbers of terms. This method belong to Perturbation theory, which is a very important theory in solving problems in applied mathematics when we are unable to find exact solutions to our equations.

REFERENCES

[1] N. G. de Brujin, "Asymptotic Methods in Analysis", *Dover publications, Inc*. New York, 1981.

[2] C. M. Bender, S. A. Orszag, "Advanced Mathematical Methods for Scientists and Engineers I: Asymptotic methods and perturbation theory", *Springer science & Business Media*, 2013.

[3] C. J. Chapman, H. P. Wynn, "Fractional power series and the method of dominant balance", *the royal society publishing*, 2021.

[4] W. Wasow, "Asymptotic Expansion for ordinary differential equations", *Dover publications, Inc*. New York, 1993.

[5] J. Callaham, J. V. Koch, B. W. Brunton, J. N. Kutz, S. L. Brunton, "Learning dominant physical processes with data-driven balance models", nature communications, 2021.

[6] E. Fan, H. Zhang, "A note of the homogenous balance method", *physics letters A*, vol. 246, issue 5, pp.403-406, 1998.

[7] S. A. El-Wakil, E. M. Abulwafa, A. Elhanbaly, M. A. Abdou, "The extended homogenous balance method and its applications for a class of nonlinear evolution equations", *Chaos, solitons & fractals*, vol. 33, issue 5, pp. 1512-1522, 2007.

[8] M. Wang, Y. Zhou, Z. Li, "Application of a homogenous balance method to exact solutions of nonlinear equation in mathematical physics", *physics letters A,* vol. 216, issue 5, pp. 67-75, 1996.

[9] G. D. Birkhoff, "On a simple type of irregular singular point", *Transactions of the American Mathematical Society*, vol. 14, issue 4, pp. 462-476, 1913.

[10] A. H. Nayfeh, "Asymptotic solutions of linear equations" in "*Perturbation methods*", USA: John Wiley-VCH Verlag GmbH & Co. KGaA, 2004, pp.309-312 .

[11] A. H. Nayfeh, "Linear Equations with variable coefficients" in "*Problems in perturbation*". USA: John Wiley & Sons Inc. 1985, pp.399-410.

[12] A. H. Nayfeh, "*Introduction to Perturbation Techniques*", New York, USA: John Wiley & Sons, 2004.