TYBES OFRESULTANT MATRICES FOR MULTIGRADED POLYNOMIAL SYSTEMS

Mariam R. Abusetta*, Najla J. Alawis, Mohamed M. Alargat

Mathematics Department, Science Faculty, Al-Asmarya University, Zliten, Libya
** Corresponding author:abosettamariam@gmail.com*

ABSTRACT

Resultant computation eliminates variables is an important tool to answer posed by the given polynomial system. Dating back from as much as 200 years ago, it has become a classical algebraic tool to determine whether a given polynomial system has a common root without explicitly solving for the roots. The study of resultant goes back to the classical work of Bezout, Sylvester, Cayley, Macaulay and Dixon.

1. INTRODUCTION

Application of polynomial in various fields stress the fundamental significance of solving and manipulating polynomial equations efficiently. In addition polynomials are often used as basic objects to solve certain problems that can be modeled using polynomial equations. Given a system of polynomial equations leads to some other simplified form of the equations with less variables computed to the original system, a sequence which is called resultant of the system.

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2. TYPE OF RESULTANT MATRICES.

1. Cayley and Bezout Resultant Matrix

In 1848, Caylay defined the resultant via a series of n alternating divisions of determinants and more recently his formula was related to Koszul complex associated to the given polynomial as mentioned by Chardin.

Definition1:

Let f(z) and g(z) be two complex polynomials of degree at most n with coefficients (note that any coefficient could be zero) (3)

$$f(z) = \sum_{i=0}^{n} u_i z^i , \;\; g(z) = \sum_{i=0}^{n} v_i z^i$$

The Bezout matrix of order n associated with the polynomials f and g is Bn(f,g)=(bij), i,j=1,2…n, where the coefficients result from theidentity.

$$\frac{f(x)g(y) - f(y)g(x)}{x - y} = \sum_{i,j=1}^{n} b_{ij} x^{i-1} y^{j-1}$$

It is in $\mathbb{C}^{n\times n}$and the entries of that matrix are such that if we note for each *i,j=1…..n,* $m_{i,j}$= *min{i, n+1-j},* then

$$b_{ij} = \sum_{k=1}^{m_{ij}} u_{j+k-1} v_{i-k} - u_{i-k} v_{j+k-1}$$

To each Bezout matrix, one can associate the following bilinear form called Bezoultion.

Bez: $\mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$: $(x, y) \to Bez(x, y) = x^* B_n(f, g) y$.

Example 1: For *n=3*, we have for any polynomial f and g of degree (at most) 3:

$$B_3(f,g) = \begin{bmatrix} u_1v_0 - u_0v_1 & u_2v_0 - u_0v_2 & u_3v_0 - u_0v_3 \\ u_2v_0 - u_0v_2 & u_2v_1 - u_1v_2 + u_3v_0 - u_0v_3 & u_3v_1 - u_1v_3 \\ u_3v_0 - u_0v_3 & u_3v_1 - u_1v_3 & u_3v_2 - u_2v_3 \end{bmatrix}$$

Let $f(x) = 3x^3 - x$ and $g(x) = 5x^2 + 1$ be two polynomials.Then

$$B_u(f,g) = \begin{bmatrix} -1 & 0 & 3 & 0 \\ 0 & 8 & 0 & 0 \\ 3 & 0 & 15 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The last row and column are zero as *f, g* have degree. Strictly less than n (equal 4). The other zero entries are because for each $i=0,\ldots n$, either u_i or v_i is zero.

Properties

- $B_n(f,g)$ is symmetric (as a matrix).
- $B_n(f,g) = -B_n(g,f)$
- $B_n(f,f) = 0$
- $B_n(f,g)$ is bilinear in (f,g)
- $B_n(f,g)$ is in $R^{n\times n}$ if *f* and *g* have real coefficients;
- $B_n(f,g)$ is nonsingular with $n = max(deg(f), deg(g))$ if and only if *f* and *g* have no common roots.
- $B_n(f,g)$ with $n = max(deg(f), deg(g)$ has determinant which is the resultant of *f* and *g*.

2. Sylvester Resultant Matrix

The resultant of two homogeneous polynomials was first studied by Euler and Bezout although it bears the name of Sylvester. He defined it as the determinant of a matrix whose entries are either zero or some polynomial coefficients.Whenever the resultant is expressible as the determinant of a single matrix. We have a Sylvester- type formula. For a few case with n<=6 Sylvester type formula is given by Jouunolos Morley and Coble [2].In 1853, Sylvester resultant was as the determinant of a matrix in the coefficient of the polynomial system.The Sylvester resultant matrix *SyL(f,g)* for two univariate polynomials in one unknown is well known in elimination polynomials in one unknown is well known in elimination theory.

If$P(z) = \sum_{i=0}^{m} a_i z^i$, $q(z) = \sum_{i=0}^{m} b_i z^i$, then the Sylvester matrix associated to $P(z)$ and $q(z)$ is the $(n+m) \times (n+m)$ matrix obtained as follows.

- The first row is $(P_m, P_{m-1}, \ldots P_1, P_0, \ldots 0)$
- The second row is the first row, shifted one column to the right the first element of the row is zero.
- The following $(n-2)$ rows are obtained the same way, still filling the first column with a zero.
- The $(n+1)$-*th* row is : $(q_n, q_{n-1}, \ldots q_1, q_0, \ldots 0)$

The following rows are obtained the same way as before. Therefore if we substitute $m=3$, $n=2$, the matrix is

$$S_{P,q} = \begin{bmatrix} P_3 & P_2 & P_1 & P_0 & 0 \\ 0 & P_3 & P_2 & P_1 & P_0 \\ q_2 & q_1 & q_0 & 0 & 0 \\ 0 & q_2 & q_1 & q_0 & 0 \\ 0 & 0 & q_2 & q_1 & q \end{bmatrix}$$

Example 2: Given the polynomial as follow:

$P_1(x)= a_3x^3+a_2x^2+a_1x+a_0$

$P_2(x)=b_2x^2+b_1x+b$ The Sylvester resultant matrix for P_1 and P_2 is:

$$\begin{bmatrix} a_3 & a_2 & a_1 & a_0 & 0 \\ 0 & a_3 & a_2 & a_1 & a_0 \\ b_2 & b_1 & b_0 & 0 & 0 \\ 0 & b_2 & b_1 & b_0 & 0 \\ 0 & 0 & b_2 & b_1 & b \end{bmatrix}$$

The determinant of the Sylvester resultant matrix of these two polynomials is the resultant of the polynomials. The Sylvester matrix also leads to the formulation of the Sylvester Dialytic matrix whose determinant also gives the resultant. Sylvester Dialytic matrix can be obtained for system *m,s* of bivariate polynomials.

3. Sylvester Matrix Dialytic Method

The Sylvester resultant matrixfor univariate case is well known and is implemented in all major computer algebra systems. It is the transpose of the matrix obtained Euler's construction and is computed using the so called dialytic method.

Consider monomial sets$X= \{x^{m-1}, x^{m-2}, \dots\ x^2, x, 1\}$,$Y =\{ x^{n-1}, x^{n-2}, \dots\ x^2, x, 1\}$and let $X.f = \{x^i.f : x^i \in X\}$ and $Y.g = \{x^i.g : x^i \in Y\}$be the corresponding polynomial sets.Thenthe matrix representing$\begin{pmatrix} X & f \\ Y & g \end{pmatrix} =$

$$\begin{pmatrix} X^{m-1}f \\ X^{m-2}f \\ \vdots \\ f \\ X^{n-1}g \\ X^{n-2}g \\ \vdots \\ g \end{pmatrix} = \begin{pmatrix} a_n & a_{n-1} & \cdots & a_0 & & & \\ & a_n & a_{n-1} & \cdots & a_0 & & \\ & & & & & & \\ & & a_n & a_{n-1} & \cdots & a_0 & \\ b_m & b_{m-1} & \cdots & b_0 & & & \\ & b_m & b_{m-1} & \cdots & b_0 & & \\ & & & & & & \\ & & b_m & b_{m-1} & \ldots & b_0 & \end{pmatrix} \times \begin{pmatrix} X^{n+m-1} \\ \vdots \\ X^n \\ \vdots \\ X^m \\ \vdots \\ X \\ 1 \end{pmatrix}$$

Is called Sylvester matrix.

Example 3: (Sylvester matrix of quadratic and linear polynomial).

Consider a polynomial system of two univariate polynomials of degree 2 and 1 respectively.

$$\mathcal{F} = \begin{cases} f_0 = ax^{2+bx+c} \\ f_1 = Sx + t \end{cases}$$

The Sylvester matrix for the above system is

$$\grave{\mathcal{F}} = \begin{pmatrix} f_0 \\ f_1 \\ xf_1 \end{pmatrix} = \underbrace{\begin{pmatrix} a & b & c \\ 0 & s & t \\ s & t & 0 \end{pmatrix}}_{S} \times \begin{pmatrix} x^2 \\ x \\ 1 \end{pmatrix},$$

and

$$\det S \quad \begin{aligned} &= at^2 - sbt + cs^2 \\ &= \left(\left(-b + \sqrt{b^2 - 4ac}\right)s + 2at\right)\left(\left(-b - \sqrt{b^2 - 4ac}\right)s + 2at\right) \\ &= a\left(\frac{-b+\sqrt{b^2-4ac}}{2a}s + t\right)\left(\frac{-b-\sqrt{b^2-4ac}}{2a}s + t\right) \end{aligned}.$$

Recently, the Sylvester matrix formulation has been generalized to polynomials in more than two variables. A matrix whose entries are either zero or the coefficients of polynaila system so that the resultant is expressible as it determinant is called the Sylvester – typed matrix and the resultant formulation is a Sylvester type formula. For a few cases when the number of variables are less than or equals 6, a Sylvester type formula is given by Jounolou, Morley and Coble [5].

4. Macaulay Resultant Matrix.

Macaulay (1902), Macaulay (1921)- resultant as quotient for but not very helpful. Difficulty overcomes by Dixon. Macaulay gives the most compact expression of the resultant as the quotient of a determinant by one of its minors, while Hurwitz introduce inertia forms, which are resultant multiples such that their Greatest Common Divisor equals the resultant in 1913 . In all cases, the non-zero entries of the matrices are coefficients of th e given polynomials. Various more recent algorithms exist to construct this resultant. This expression has been approved for the case when all polynomials have the same degree. The earliest of the contemporary constructive approaches was Lazard's which used the U-resultant to solve the polynomial system. Canny designed a parallelizable algorithm for computing the resultant of specialized polynomials out of inertia forms [13].

Definition of Macaulay Resultant 2:

Given n homogenous polynomial $P= \{ P_1, P_2, P_3, \ldots .P_n\}$ of degree $d_1, d_2, d_3 \ldots d_n$ in the variables $x= (x_1, x_2, \ldots x_n)$, $P(x) = \begin{bmatrix} P_1(x_1, \ldots., x_n \\ \vdots \\ P_n(x_1, \ldots., x_n \end{bmatrix}$. Macaulay Resultant Matrix of the problem is

$$M = \begin{bmatrix} a_0 & b_0 & \cdots & & & \\ \vdots & & & & & \\ & & & & a_0 & b_0 \\ a_{n-1} & b_{n-1} & & \cdots & a_1 & b_1 \\ a_n & b_n & & & & \\ & & & & \vdots & \\ & & & & a_n & b_n \end{bmatrix}$$

Submatrix N of M is $N = \left[\begin{pmatrix} a_0 & \cdots & a_n \\ \vdots & \ddots & \vdots \\ b_0 & \cdots & b_n \end{pmatrix}\right]$

The Macculay resultant is a follow: $R = \frac{\det(M)}{\det(N)}$,where *det(M)* is the determinant of matrix *M*, and *det(N)* is the determinant of the matrix *N(6)*.

Example4:let $S = \sin(t)\,, T = \tan(t)$ and $C = \cos(t)$ then

$$\mathcal{F} = \begin{cases} f_0 = C^2 + S^2 - 1 = 0, \;\; for\; (\sin(t)^2 + \cos(t)^2 = 1) \\ f_1 = CT - S = 0, \quad for \left(\tan(t) = \sin(t)/_{\cos(t)^2}\right) \\ f_2 = x - aS = 0, \\ f_3 = y - aT(1+S) = 0. \end{cases}$$

We wish to eliminate , S and T to get a single equation in terms of x, y and a.Note that the total degree of polynomials $\{f_0, f_1,\ f_2,\ f_3\}$ are$\{\,2, 2,\ 1,\ 2\,\}$, and therefore, $D = 4$.The size the monomial set X is $|X| = \binom{D+d}{d} = \binom{7}{3} = 35$. Hence, $M_{\mathcal{F}`}$ is will have 35 rows but because a lot of powers are "missing" , the matrix has 32 columns.

The monomials used to construct the matrix are

$X_0 = \{1, C^2, S^2, TC, ST, SC, T^2, T, C, S\}$, $|X_0| = 10$,

$X_1 = \{1, C^2, TC, ST, SC, T^2, T, C, S\}$, $|X_1| = 9$,

$X_2 = \{1, C^2, TC, ST, SC^2, TC^2, C^3, SC, T, C, S, STC\}$, $|X_2| = 12$,

$X_0 = \{1, ST, T, S\}$,
$|X_0| = 10$,

The fact that the matrix is very easy to compute, as well as that it can be stored quite efficiently, the main difficulties with Macaulay matrix are:

1. Matrix size , and therefore high complexity and huge extraneous factors.
2. Computes resultant over , where in most particular cases, the affine space of interest.
3. In most cases, the matrix is singular. Even though it is not clear if a maximal minor containing the resultant always exist.

Example5 : Solve the intersection points of the following circle P_1 and ellipse P_2 :

$$P_1 = x_1^2 + x_2^2 - 2 = 0$$
$$P_2 = x_1^2 + 6x_2^2 - 3 = 0$$

Solution:

The system above is solved by the u-resultant method. The computation of the u-resultant is a standard tool for finding all isolated roots. A related approach reduces the problem to solve a single equation in one variable, then lifting these solutions to the common roots of the original system.

Let x_3 be a new variable and then convert P_1 and P_2 into homogeneous system

$$\widehat{P_1} = x_1^2 + x_2^2 - 2x_3^2 = 0$$
$$\widehat{P_2} = x_1^2 + 6x_2^2 - 3x_3^2 = 0$$

Let $P_u = u_1x_1 + u_2x_2 + u_3x_3$, where u_1, u_2, u_3 are new variables. The u-resultant of $\{P_1, P_2\}$ in variables x_1 and x_2 is the Macauly resultant of polynomial systems $\widehat{P_1}, \widehat{P_2}$ and P_u in variables x_1, x_2 and x_3 .

The Macaulay matrix of the problem is

$$M = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & -2 \\ 1 & 0 & 0 & 6 & 0 & -3 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 6 & 0 & -3 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 6 & 0 & -3 \\ 0 & u_1 & 0 & u_2 & u_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & u_1 & 0 & u_2 & u_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & u_1 & 0 & 0 & u_2 & u_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & u_1 & 0 & 0 & u_2 & u_3 \end{bmatrix}$$

The submatrix$N$ of M is:

$N = \begin{bmatrix} 1 & 1 \\ 1 & 6 \end{bmatrix}$. The Macaulay resultant is as follows:

$$R = 81u_1^4 - 18u_1^2u_2^2 - 90u_1^2u_3^2 - 10u_3^2u_2^2 + 25u_3^4 + u_2^4$$

$$R = \left(-3u_1 + u_2 + u_3\sqrt{5}\right)\left(-u_2 + u_3\sqrt{5} + 3u_1\right)\left(3u_1 + u_2 + u_3\sqrt{5}\right)\left(-u_2 + u_3\sqrt{5} + 3u_1\right)$$

Thus, we can get the coordinates of the intersection points as follows:

$$\left(\frac{3}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right), \left(\frac{3}{\sqrt{5}}, -\frac{1}{\sqrt{5}}\right), \left(-\frac{3}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right), \left(-\frac{3}{\sqrt{5}}, -\frac{1}{\sqrt{5}}\right).$$

5. Dixon Resultant Matrix.

Dixon proposed a matrix called the Dixon matrix whose determinant is a multiple of the resultant. In it is classical form, this method works only on a small class of polynomial systems known as generic bi-degree systems

Definition 3:Dixon polynomial

$$\theta_{x,y}(f_0, f_1, f_2) = \frac{1}{(\bar{x} - x)(\bar{y} - y)} \begin{vmatrix} f_0(x,y) & f_1(x,y) & f_2(x,y) \\ f_0(\bar{x},y) & f_1(\bar{x},y) & f_2(\bar{x},y) \\ f_0(\bar{x},\bar{y}) & f_1(\bar{x},\bar{y}) & f_2(\bar{x},\bar{y}) \end{vmatrix}$$

Where $\bar{x}$ and $\bar{y}$ are new variables.Let$\bar{X}$ be an ordered set of all monomials appearing in $\theta(f_0, f_1, f_2)$ in terms of variables $\bar{x}$, $\bar{y}$ and X be the set of all monomials in terms of variables x and y. Then $\theta_{x,y}(f_0, f_1, f_2) = [\ldots \bar{x}\bar{y} \ldots]\Theta_{x,y}[\ldots xy \ldots]\theta_{x,y}(f_0, f_1, f_2) = \bar{X}\Theta_{x,y}\mathrm{X}$, Where $\Theta_{x,y}$ is called the Dixon matrix. Note that $\Theta_{x,y} = \Theta_{y,x}^{\mathrm{T}}$, where the order of variables x, y is reversed. We will drop variable subscripts since it suffices to consider any variable order. The monomial support of a general bi-degree (m, n)polynomial in (x, y) is

$$A_{m,n=}\ \{0,1,\ldots,m\}\times\{0,1,\ldots,n\}=0..m\times 0..n$$

The classical Dixon resultant is the determinant $|\Theta_{x,y}|$ when$A = A_{m,n}$. The row an column supports of the classical Dixon matrix are

$\bar{X}_{m,n} = 0\,..\,m - 1 \times 0\,..2n - 1,$

$X_{m,n} = 0\,..2m - 1 \times 0\,..\,n - 1,$

Since the set cardinalities $|\bar{X}_{m,n} = X_{m,n}| = 2mn$, the order of the classical Dixon matrix $\Theta_{x,y}$ is $2mn$.

Example 6: Consider the Dixon polynomial

$$\theta_{x,y}(f_0, f_1, f_2) = -4z^2 + (z^2 - 5z)x + (z^2 - 5z)\bar{x} - 4x\bar{x} + (z^2 - z)y + (z^2 - z)\bar{y} - 4y\bar{y}$$

We can write $\theta_{x,y}$ in matrix form by extracting the column support first,

$$\theta_{x,y} = [-4z^2 + (z^2 - z)\bar{y} + (z^2 - 5z)\bar{x}(z^2 - z)y + (z^2 - z) - 4\bar{y} + 0 \qquad (z^2 - 5z) + 0 - 4\bar{x}]\begin{bmatrix}1\\ y\\ x\end{bmatrix}$$

Then we extract the row support and become

$$\theta_{x,y} = [1 \quad \bar{y} \quad \bar{x}]\begin{bmatrix} -4z^2 & z^2 - z & z^2 - 5z \\ z^2 - z & -4 & 0 \\ z^2 - 5z & 0 & -4 \end{bmatrix}\begin{bmatrix} 1 \\ y \\ x \end{bmatrix}$$

The Dixon resultant is the determinant of $\theta_{x,y}$ that is $|\Theta_{x,y}| = 40z^2 + 8z^4 - 48z^3$. if $\mathcal{F} = \{f_0, f_1, f_2\}$ has a common zero, it is also of for any value of new variables x and y. Thus $\Theta \times \mathrm{X} = 0$, whenever x and y are replaced by a common zero of f_0, f_1, f_2.

6. Sparse Resultant Matrix.

The Sparse Resultant generalizes the classical homogeneous resultant and exploit the structure of the given polynomials. Its size depends only on the geometry of the input Newtonpolytopes. Sparse elimination theory generalizes several results of classical elimination theories on Multivariatepoolynami;a systems by considering the structure of the given polynomials, namely the coefficient which are a priory zero and the support and Newton polytopes defined by the nonzero coefficients.

The Sparse resultant only considers affineroots and generalizes the classical resultant of n+1 homogeneouspolynomials in n+1 variables in the sense that day coincide when all polynomial s coefficient are non zero. The sparse resultant coincides with Sylvester resultant If the system comprised two uivariate polynomial [6].

The sparse resultant for multivariate polynomial is generalization of the Sylvester resultant. The sparse resultant leads to a stronger algebraic and combinatorial results since sparseity properties can be exploited in the ters of the structure of Newton polytopes of the systems under considerations. IN addition the structure of the sparse resultant matrix can also be further exploited when reducing the matrix into an eigen value decomposition for solving polynomialssystems

3. CONCLUSION

Thispaper provides an overview of resultant matrices and classical work of Bezout, Sylvester, Caylay, Macaulay, Dixon and Sparse resultant matrix.

In addition, the methods the formulating and constructing Sylvester type matrix for bivariate polynomial system will also be investigated before the generalization to multivariate polynomials in n variables can be employed.

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