An Efficient Finite Element for Vibration Analysis of Symmetric Sandwich Beams Subjected to Harmonic Bending Excitations

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Abstract

An efficient sandwich beam finite element is developed for the coupled axial bending vibration analysis of sandwich beams subjected to general harmonic bending excitations. A Hamilton’s variational formulation is used to derive the governing field equations, which are exactly resolved to establish the exact solution for dynamic response in steady state form. A set of shape functions is created using the exact solution of the governing equations. These functions are employed to construct a finite element for beams. This finite element features two nodes, each with six degrees of freedom, effectively representing the coupling between the extensional and flexural behaviours of symmetric sandwich beams subjected to harmonic bending loads in static and steady-state dynamic responses. To establish the exactness and effectiveness of the current sandwich beam element, it is compared with established Abaqus finite element solution and other solutions reported in the literature. The newly developed sandwich beam element demonstrates freedom from discretization errors observed in alternative interpolation methods. It produces results that closely match those obtained from other finite element solutions, but at a significantly reduced computational and modelling cost.

Keywords: An efficient finite element; exact shape functions; harmonic forces.

INTRODUCTION AND OBJECTIVE

Structural beams and panels with a sandwich construction are of great interest because of their advantages of combining light weight and high bending stiffness and strength, transverse shear stiffness, high damping capacities, excellent thermal insulation properties, and excellent energy absorption capability. A typical sandwich structural beam consists of two face layers separated by a lightweight thick core layer such as foams or honeycombs. The sandwich structures are used widely in aerospace, marine, automotive industries and civil infrastructure
applications. The fundamental idea behind the sandwich beam involves the top and bottom layers bearing the loads generated by bending (including flexural load and compression), while the core layer carries the transverse loads induced by shearing.

While research efforts over recent years have primarily focused on analysing the vibration of sandwich beams using different beam theories under various dynamic forces, the majority of previous investigations have primarily focused on analysing the natural vibrations of these beams. Many researchers have offered analytical solutions employing different beam theories to characterize the free vibration response of sandwich beams. Banerjee [1] employed an analytical method to establish the dynamic stiffness matrix technique for a three-layered symmetric sandwich beam. In this formulation, the sandwich faces are modelled using the Euler-Bernoulli beam model, allowing for axial extension, while it's presumed that the core layer undergoes only lateral shear. Banerjee and Sobey [2] experienced improvement of the earlier sandwich model in [1], by representing the top and bottom face layers as Rayleigh beams, while the bending of the core layer was governed by the Timoshenko beam theory. Howson and Zare [3] utilized the exact dynamic matrix to explore the natural bending vibration of three-layered sandwich beams with unequal face layers. Ghugal and Shikhare [4] applied the trigonometric shear deformation beam theory to investigate the bending characteristics of sandwich beam. Rajesh and Kumar [5] investigated the natural oscillation characteristics of different sandwich beams with viscoelastic properties subjected to different boundary conditions. Sayyd and Ghugar [6], employed the trigonometric shear deformation theory to examine the impact of shear deformation in the static bending analysis of sandwich beams having soft-cores. In their approach, they (a) utilized the principle of virtual work to formulated the field equations and boundary conditions, and (b) applied the Navier solution method, to derive the closed-form solutions for sandwich simply supported beams. Zare etal. [7] developed the dynamic stiffness matrix to evaluate the combined bending and longitudinal free vibration of sandwich beams. Dorostghoal et al. [8] derived the dynamic equations for three-layered sandwich curved beams having symmetric faces by utilizing the variational form of Hamilton’s principle. In their formulation, the sandwich face layers are considered to behave like Euler-Bernoulli beams whereas the core layer deforms only in shear. They solved the governing field coupled equations through closed form solution to develop an exact dynamic stiffness matrix of a curved sandwich beam. In thier study, the natural frequencies of curved sandwich beams are computed by applying the well-known algorithm of Wittrick-Williams. Recently, Hjaji et al. [9] investigated the vibration response of symmetric sandwich beams subjected to harmonic excitations. The coupled equations and related boundary conditions are determined from the Hamiltonian variational principle. In their formulation, the sandwich faces are modelled as Euler-Bernoulli beam while the core layer allows only shear deformation. The exact expressions for symmetric sandwich beams having different boundary constraints are exactly obtained by solving the governing field equations. Their formulation in [9] demonstrates the efficiency of the closed form solutions by comparison with various published results. More recently, Lee [10] employed the transfer matrix method to investigate the natural vibration response of symmetric sandwich beams having soft cores, in which the mass of the core layer is neglected, while the coupling between the axial and bending deformations is considered. In his formulation, the Euler-Bernoulli beam theory is used to model the top and bottom sandwich faces, although the shear deformation is only allowed in the core layer.

Several researchers have examined the dynamic behaviour of sandwich beams through finite element analysis. Ahmed [11,12] employed the finite element approach to investigate
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the bending oscillation of curved sandwich beams having elastic cores. Baber et al. [13] formulated a finite element model to investigate the harmonic response of the sandwich beams under harmonic excitations with thin and thick viscoelastic core layers. Their model presumes that the sandwich face layers are elastic and behave as Bernoulli-Euler beams, neglecting shear deformation. Their finite element formulation uses a core displacement varied nonlinearly along the sandwich beam and through the sandwich core thickness. A simple approximation for the core's response involves expressing all core variables in terms of the face plate displacements. Their finite element model employed conventional beam shape functions to construct a twelve-degree-of-freedom element. Hashem and Adique [14] developed the dynamic finite element method based on the weak integral form of the governing differential equations to establish the free vibration of sandwich beams with symmetric face layers. Their formulation assumed that the sandwich face layers assumed to behave according to Euler–Bernoulli beam theory in which the effects of rotary inertia in ignored, while the sandwich core undergoes shear deformation only. Based on Reissner’s mixed variational principle, Bouziane et al. [15] used a mixed interface finite element to analyse the sandwich beams. In their approach, they devised a 7-node two-dimensional mixed finite element, comprising 5 displacement nodes and 2 stress nodes. This mixed interface finite element was achieved by sequentially employing the delocalization technique and the static condensation procedure. The element ensures the continuity of transverse stress and displacement fields at the interface.

Sulbhewar and Ravendranath [16] developed an efficient piezoelectric sandwich beam element to explore coupled polynomial interpolation schemes for shear mode sandwich beams. Based on the Hamilton variational principle, they derived the governing coupled equations, establishing relationships between field variables. Subsequently, they derived a sandwich beam theory-based finite element model with coupled polynomial expressions for analysing shear mode sandwich beams from these governing coupled equations. They validated their finite element model for static and modal analyses by comparing numerical results with those obtained from other finite element solutions. Cortes and Sarría [17] presented the dynamic analysis of three-layer sandwich beams featuring constrained layer damping with a thick viscoelastic layer, where the shear effect is considered negligible. Their finite element model was utilized to investigate the dynamic behaviour of sandwich beams in a constrained damping configuration. Two-dimensional finite element models, accounting for extensional and shear stresses as well as longitudinal, transverse, and rotational inertias, were discretized into bilinear quadrilateral elements with four nodes under plane-stress conditions. Huang et al. [18] developed an efficient finite element model, based on the first-order shear deformation theory and Hamilton variational principle, incorporated with the Biot model for analysing the vibration of sandwich composite beams with a viscoelastic material core between elastic faces. Al-Itbi et al. [19] achieved the structural discretization of viscoelastic sandwich beams by using a two-node composite beam element with four degrees of freedom. In their formulation, the frequency-dependent viscoelastic dynamics of the sandwich beam were established through finite element analysis and experimental validation. They analyzed the static response of sandwich beams subjected to distributed forces along the beam axis. Their sandwich core is composed of a functionally graded porous material, while the top and bottom faces are constructed from isotropic homogeneous materials. They utilized ANSYS program in their study by using a finite beam element based on the first-order shear deformation theory. Their formulation incorporates
the influence of the porosity coefficient, boundary conditions, and the type of porous material on the static response.

The finite element formulations based on approximate interpolation shape functions involve spatial discretization errors, and consequently require fine meshes to converge to the accurate results. In contrast, the present finite element formulation based on exact solutions offers two advantages: (a) they mitigate discretization errors that arise in traditional interpolation schemes and converge towards exact results with a minimal number of degrees of freedom; and (b) they result in beam elements that are devoid of shear locking. Within this context, the aim of this paper is to create an exact and efficient finite element model for sandwich beams, featuring six degrees of freedom per element. The new element is used to investigate the coupled axial-bending dynamic analysis of sandwich beams with symmetric sandwich faces subjected to harmonic bending forces. The finite element formulation being pursued relies on shape functions that precisely adhere to the coupled field equations. It is designed to capture both quasi-static and steady-state dynamic responses and accurately compute the natural frequencies and vibration modes of sandwich beams.

MATHEMATICAL FORMULATION

In a rectangular Cartesian coordinate system, Figure 1 shows a structural sandwich beam of span $L$ under harmonic bending forces. The total thickness of the sandwich cross-section is $h$, $h_f$ and $h_c$ are the thicknesses of the face and core layers, respectively, (both faces have identical thicknesses, $h_{f1} = h_{f2} = h_f$), and $b$ is the width of the cross-section. The top and bottom layers are considered as thin faces and are represented using the Euler-Bernoulli beam theory, incorporating longitudinal and bending rigidity. Meanwhile, the core layer consists of lightweight material (i.e., core mass is ignored compared to mass of the sandwich faces), and considers only the shear deformation, i.e., the longitudinal stresses in the core layer are assumed zero.

All layers of the structural sandwich beam are fully bonded, i.e., these layers exhibit no delamination during deformation. Additionally, the material composing the sandwich faces is significantly stiffer than the core material. In this formulation, it is assumed that both the face and core materials are homogeneous, isotropic, and linearly elastic, with all displacements and strains considered small. When subjected to a harmonic bending forces, the sandwich beam deforms only in $x$-$z$ plane. All three layers are supposed to experience the same lateral deformation, denoted as $w(x,t)$, while the longitudinal displacements of the both faces are represented as $u_{f1}(x,t)$ and $u_{f2}(x,t)$, respectively. This symmetry in the motion of the sandwich beam requires that both sandwich faces have the same thickness, i.e., $u_{f1}(x,t) = -u_{f2}(x,t) = u(x,t)$.

Figure 1: Three-layered sandwich beam under harmonic forces
Total Potential Energy Expression

The total potential energy of the layered sandwich beam consists of three parts, the strain energy of the both sandwich faces, the strain energy of the core, and the potential energy of the applied dynamic forces, i.e., \( \Pi = U_f + U_c + V_f \).

The internal strain energy due to normal stresses in both sandwich faces is given as [1,9]:

\[
U_f = U_{f1} + U_{f2} = \int_0^L E_f A_f (u')^2 \, dx + \int_0^L E_f I_f (w'')^2 \, dx
\]

where \( E_f \) is the modulus of elasticity of the sandwich face, \( A_f \) is the cross-sectional area of the sandwich face, \( I_f \) is the moment of inertia of the sandwich face, \( u(x, t) \) and \( w(x, t) \) are the axial and transverse displacements, respectively.

The internal strain energy in core layer due to shearing stress is [1,9]:

\[
U_c = \frac{1}{2} \int_0^L k_s G_c A_c \left[ \left( 1 + \frac{h_f}{h_c} \right) w'(x, t) + \frac{2}{h_c} u(x, t) \right]^2 \, dx
\]

where \( G_c \) is the rigidity modulus of the core, \( A_c \) is the sandwich core cross-sectional area, and \( k_s \) is the correction factor of shear distribution.

Subsequently, the combined strain energy of the sandwich beam resulting from both strains can be expressed as:

\[
U = \int_0^L \left( E_f A_f (u')^2 + E_f I_f (w'')^2 + \frac{k_s G_c A_c}{2} \left[ \left( 1 + \frac{h_f}{h_c} \right) w' + \frac{2}{h_c} u \right]^2 \right) \, dx
\]

and, the potential of the applied bending forces is expressed by:

\[
V_f = \int_0^L \left[ q_x(x, t)w(x, t) + q_z(x, t)u(x, t) + m_x(x, t)w'(x, t) \right] dx + \left[ F_x(x_e, t)w(x_e, t) + F_z(x_e, t)u(x_e, t) + M_x(x_e, t)w'(x_e, t) \right]_0^L
\]

where, \( q_x(x, t), q_z(x, t) \) are the distributed axial and transverse harmonic forces, respectively, \( m_x(x, t) \) is the distributed harmonic bending moment, \( F_x(x_e, t), F_z(x_e, t) \) are the concentrated axial and transverse harmonic forces, respectively, and \( M_x(x_e, t) \) are the bending moments at the beam ends, i.e., \( x_e = 0 \) and \( L \).

Then, the total potential energy of the sandwich beam is obtained by adding equation (3) to equation (4), as:

\[
\Pi = \int_0^L \left[ E_f A_f (u')^2 + E_f I_f (w'')^2 + \frac{k_s G_c A_c}{2} \left[ \left( 1 + \frac{h_f}{h_c} \right) w' + \frac{2}{h_c} u \right]^2 + q_x(x, t)w(x, t) \right. \\
+ \left. q_z(x, t)u(x, t) + m_x(x, t)w'(x, t) \right] dx + \left[ F_x(x_e, t)w(x_e, t) + F_z(x_e, t)u(x_e, t) + M_x(x_e, t)w'(x_e, t) \right]_0^L
\]
Kinetic Energy Expression

Under the assumption that the sandwich core of the sandwich is composed of lightweight material, making its mass negligible compared to the mass of the face layers, the expression of kinetic energy can be defined as:

\[ T = \int_0^l \rho_f A_f [(\dot{u})^2 + (\dot{w})^2] \, dx \]  

...(6)

where \( \rho_f \) is the material density of the sandwich face.

Expressions for Applied Forces and Displacement Functions

The various distributed axial and bending harmonic forces within the sandwich beam are assumed to have the following forms:

\[ q_x(x, t), q_x(x, t), m_x(x, t) = [\vec{q}_x(x), \vec{q}_x(x), \vec{m}_x(x)] \quad e^{i\Omega t} \]  

(7)

and the concentrated harmonic forces and moments at both ends of the sandwich beam:

\[ F_x(x, t), F_x(x, t), M_x(x, t) = [\vec{F}_x(x), \vec{F}_x(x), \vec{M}_x(x)] \quad e^{i\Omega t} \]  

(8)

where \( \Omega \) is the forcing frequency of the harmonic excitation, and \( i = \sqrt{-1} \).

Given the harmonic excitations, the displacement expressions corresponding to steady-state component of the response are assumed to adopt the following form:

\[ u(x, t), w(x, t) = [U(x), W(x)] \quad e^{i\Omega t} \]  

(9)

where \( U(x) \) and \( W(x) \) are the amplitude space functions for axial and transverse displacements, respectively. Equation (9) is designed to only capture the coupled response in steady-state dynamics of the sandwich beam. In other words, the displacements postulated in equation (9) disregard the transient part of the sandwich beam’s response.

Governing Field Equations

By employing the Hamilton’s variational principle,

\[ \delta \int_{t_1}^{t_2} (T - \Pi) \, dt = 0, \quad \text{for } \delta u(x, t)|_{t_1}^{t_2} = \delta w(x, t)|_{t_1}^{t_2} = 0 \]  

(10)

where \( \delta \) is defined as the variational operator, \( t_1 \) and \( t_2 \) are the interval of time.

By substituting equations (7-9) into energy expressions (5-6) and subsequently into Hamilton’s principle equation (10), the resulting equations yield the field equations for the extensional-bending coupled response of sandwich beams subjected to dynamic excitations as follows:

\[
\begin{bmatrix}
T_{11} & -T_{12} \\
T_{12} & T_{22}
\end{bmatrix}_{2 \times 2}
\begin{bmatrix}
U(x) \\
W(x)
\end{bmatrix}_{2 \times 1} =
\begin{bmatrix}
\vec{q}_x(x) \\
\vec{q}_x(x)
\end{bmatrix}_{2 \times 1}
\]  

(11)

where \( T_{11} = 2E_f A_f \frac{d}{dx}^2 + (2\rho_f A_f \Omega^2 - 4k_G \eta_s), \)
\( T_{12} = 2\eta_s (h_f + h_c) \frac{d}{dx}, \)
\( T_{22} = (-2E_f I_f \frac{d}{dx}^4 + \eta_s (h_f + h_c)^2 \frac{d}{dx}^2 + 2\rho_f A_f \Omega^2), \)
\( \eta_s = \frac{k_G}{h_c^2}, \)
\( \frac{d^n}{dx^n}, \)
\( n = 1, 2. \)
Exact Closed Form Solution of Field Equations

The homogeneous solution of the governing coupled field equations in equation (11) is derived by setting the loading terms in the field equations to zero, i.e., $q_x(x) = q_z(x) = 0$. The spatial displacement functions are assumed to follow the exponential form:

$$\langle x(x) \rangle_{1\times2} = \langle U(x) \rangle_{1\times2} = \langle A_i \rangle_{1\times2} e^{\beta_i x}$$

(12)

where $\langle A_i \rangle_{1\times2}$ is a vector of unknown integration constants corresponding to roots $\beta_i$. From the displacement fields postulated in equation (12), by substituting into equation (11), one obtains the following sixth order polynomial equation:

$$p_3 \beta_i^6 + p_2 \beta_i^4 + p_1 \beta_i^2 + p_0 = 0$$

(13)

where $p_0$ via $p_3$ are constants depend on the sandwich cross-section, material properties, and exciting frequency $\Omega$ and are given as:

$$p_3 = E_f A_f l_f, \quad p_2 = E_f A_f \rho_f l_f \Omega^2 - \eta_s \left(2E_f l_f + \frac{E_f A_f(h_f + h_c)}{2}\right), \quad p_1 = -\rho_f A_f \Omega^2 \left(\frac{\eta_s}{2} (h_f + h_c)^2 + E_f l_f \right), \quad p_0 = -\rho_f A_f \Omega^2 [\rho_f A_f \Omega^2 - 2\eta_s]$$

Then, the exact homogeneous solution of the displacement functions takes the form:

$$U(x) = \sum_{i=1}^{6} \langle A_i \rangle e^{\beta_i x}, \quad W(x) = \sum_{i=1}^{6} \langle B_i \rangle e^{\beta_i x}$$

(14)

Equation (14) contains a total of twelve integration constants $\langle A_i \rangle$ and $\langle B_i \rangle$ for $i = 1, 2, 3, ... 6$, whose values are not determined. However, since only six boundary conditions are given at the ends of the sandwich beam, it is required to diminish the twelve unknown constants to six distinct boundary conditions. This is achieved by representing the set of constants $\langle B_i \rangle$ in relation to the constants from another set $\langle A_i \rangle$ through the equation $\langle B_i \rangle = \langle B \rangle \langle A_i \rangle$, where

$$\langle B \rangle = \frac{E_f A_f \beta_i^2 - 2\eta_s + \rho_f A_f \Omega^2}{\eta_s (h_f + h_c) \beta_i}$$

Therefore, the exact solutions for the axial displacement $U(x)$, transverse displacement $W(x)$, and bending rotation $\Phi_x(x)$ capture the coupling deformation between axial and flexural steady state responses are given in matrix form as:

$$\langle x(x) \rangle_{1\times2} = \langle \bar{G} \rangle_{2\times6} \langle E(x) \rangle_{6\times1} \langle A_i \rangle_{6\times1}$$

(15)

where $\langle x(x) \rangle_{1\times2} = \langle U(x) \rangle_{1\times2}, \langle E(x) \rangle_{6\times1}$ is 6x6 matrix consists of exponential functions $e^{\beta_i x}$, where $i = 1, 2, 3, ..., 6$ arranged diagonally, and $\langle A_i \rangle_{1\times6} = \langle A_1, A_2, A_3, ..., A_6 \rangle_{1\times6}$ is the array of unknown constants, and $\langle \bar{G} \rangle_{2\times6} = \begin{bmatrix} \frac{1}{\bar{G}_1} & 1 & \frac{1}{\bar{G}_2} & 1 & \cdots & 1 & \frac{1}{\bar{G}_6} \end{bmatrix}_{2\times6}$.

**FINITE ELEMENT FORMULATION**

This section formulates a two-noded finite sandwich beam element with three degrees of freedom per node. The finite element is developed to establish the coupling response between the axial and transverse deformations for dynamic steady-state analysis of sandwich beams with identical facings (Fig. 2). The new sandwich beam element of length $l_e$ depends on a set of shape functions that precisely achieve the coupled field equations presented in equation (11).
Formulating of Exact Shape Functions

The unknown constants \( \{ \mathcal{B}_i \}_{6 \times 1} \) is expressed in terms of the nodal displacements by enforcing the conditions \( U(0) = U_1 \), \( W(0) = W_1 \), \( W'(0) = \Phi_x \), \( U(l_e) = U_2 \), \( W(l_e) = W_2 \), and \( W''(l_e) = \Phi_x \). The displacement field \( \{ \chi(x) \}_{1 \times 3} \) are then represented using nodal displacements as: \( \{ d_N \}_{1 \times 6} = \langle U_1 \ W_1 \ \Phi_x \ U_2 \ W_2 \ \Phi_x \rangle_{1 \times 6} \), yielding,

\[
\{ d_N(x) \}_{6 \times 1} = \left[ \begin{bmatrix} R \end{bmatrix}_{3 \times 6} \{ E(0) \}_{6 \times 1} \right]_{6 \times 1} = \left[ \begin{bmatrix} \mathcal{A} \end{bmatrix}_{6 \times 1} = \{ \psi \}_{6 \times 1} \{ \mathcal{A} \}_{6 \times 1} \right]
\]

where \( \left[ R \right]_{3 \times 6} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ \beta_1 \mathcal{G}_1 & \beta_2 \mathcal{G}_2 & \beta_3 \mathcal{G}_3 & \cdots & \beta_6 \mathcal{G}_6 \end{bmatrix}_{3 \times 6} \). From equation (16), by substituting into equation (15), one obtains:

\[
\{ \chi(x) \}_{2 \times 1} = \left[ \mathcal{G} \right]_{2 \times 6} \{ E(x) \}_{6 \times 1} \{ \psi \}_{6 \times 1} \{ d_N \}_{6 \times 1} = \{ H(x) \}_{2 \times 6} \{ d_N \}_{6 \times 1}
\]

where \( \{ H(x) \}_{2 \times 6} = \left[ \mathcal{G} \right]_{2 \times 6} \{ E(x) \}_{6 \times 1} \{ \psi \}_{6 \times 1} \) is a matrix of 12 shape functions. The interpolation shape functions described in equation (17) precisely fulfill the homogeneous solution of the field equation and depend on factors such as the length of the sandwich, cross-sectional properties, and the frequency of the applied forces.

Energy Expressions in Nodal Displacements

The strain energy, kinetic energy and potential energy of the applied forces presented in equations (3), (4) and (6) are written in terms of nodal displacements by substituting the expressions in equation (17) to yield:

\[
\delta U = \langle \delta d_N \rangle_{1 \times 6} \int_{l_e} \left\{ \{ H(x) \}_{2 \times 2} \{ S_d \}_{2 \times 2} \{ H(x) \}_{2 \times 6} + \{ H'(x) \}_{2 \times 2} \{ S_m \}_{2 \times 2} \{ H'(x) \}_{2 \times 6} \right. \\
+ \{ H''(x) \}_{2 \times 2} \{ S_\ell \}_{2 \times 2} \{ H''(x) \}_{2 \times 6} \\
+ \{ H_r(x) \}_{2 \times 2} \{ S_r \}_{2 \times 2} \{ H_s(x) \}_{2 \times 6} \left\} \{ d_N \}_{6 \times 1} e^{i \omega t} dx \}
\]

where \( \{ S_d \}_{2 \times 2} = \text{diag.} \{ 4 \eta_s \ 0 \}_{2 \times 2} \), \( \{ S_m \}_{2 \times 2} = \text{diag.} \{ 2 E_f A_f \eta_s (h_f + h_c) \}_{2 \times 2} \), \( \{ S_\ell \}_{2 \times 2} = \text{diag.} \{ 2 E_f I_f \}_{2 \times 2} \), \( \{ S_r \}_{2 \times 2} = \text{diag.} \{ 2 \eta_s (h_f + h_c) 2 \eta_s (h_f + h_c) \}_{2 \times 2} \) and \( \{ H_s \}_{2 \times 6} = \{ H_s(x) \}_{2 \times 6} \), and \( \{ H_r(x) \}_{2 \times 6} = \{ H'_s(x) \ H_1(x) \}_{2 \times 6} \).

The variation of kinetic energy is obtained as:

\[
\delta T = - \langle \delta d_N \rangle_{1 \times 6} \int_{l_e} \left\{ \{ H(x) \}_{2 \times 2} \{ T_m \}_{2 \times 2} \{ H(x) \}_{2 \times 6} \{ d_N \}_{6 \times 1} e^{i \omega t} dx \right\}
\]

where \( \{ T_m \}_{2 \times 2} = \text{diag.} \{ 2 \rho_f A_f \ 2 \rho_f A_f \}_{2 \times 2} \).
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and, the variation of potential load vector is given as:

$$\delta V = \langle \delta d_N \rangle_{1 \times 6} \left[ \int_{0}^{t_e} \left( [H(x)]_{6 \times 2}^T \bar{Q}_d(x) + [H'(x)]_{6 \times 2}^T \bar{Q}_m(x) \right) dx + \frac{t_e}{6} \right] \left\{ \int_{0}^{t_e} \delta d_N e^{t \Omega} \right\}$$

(20)

where $\langle \bar{Q}_d(x) \rangle_{1 \times 2} = \langle \bar{q}_d(x) \rangle \bar{q}_d(x)$, $\langle \bar{Q}_m(x) \rangle_{1 \times 2} = \langle 0 \rangle \bar{m}_x(x)$, and $\langle \bar{Q}_b(x) \rangle_{1 \times 2} = \langle 0 \rangle \bar{M}_x(x)$.

Discretized Equilibrium Field Equations

From equations (18-20), by substituting into Hamilton’s principle equation (10), one obtains:

$$\left( [k_e]_{6 \times 6} - \Omega^2 [M_e]_{6 \times 6} \right)_{6 \times 6} \{d_N\}_{6 \times 1} = \{F_e\}_{6 \times 1}$$

(21)

where, the stiffness matrix for the sandwich beam element is given as:

$$[k_e]_{6 \times 6} = \int_{0}^{t_e} \left( [H(x)]_{6 \times 2}^T [S_d]_{2 \times 2} [H(x)]_{2 \times 6} + [H'(x)]_{6 \times 2}^T [S_m]_{2 \times 2} [H'(x)]_{2 \times 6} + [H''(x)]_{6 \times 2}^T [S_s]_{2 \times 2} [H''(x)]_{2 \times 6} + [H_s(x)]_{6 \times 2}^T [S_r]_{2 \times 2} [H_r(x)]_{2 \times 6} \right) dx$$

(22)

The sandwich beam element mass matrix is obtained as:

$$[M_e]_{6 \times 6} = \int_{0}^{t_e} \left( [H(x)]_{6 \times 2}^T [T_m]_{2 \times 2} [H(x)]_{2 \times 6} \right) dx$$

(23)

and, the vector of potential energy load for the sandwich beam element is given as:

$$\{F_e\}_{6 \times 1} = \int_{0}^{t_e} \left( [H(x)]_{6 \times 2}^T \bar{Q}_d(x) + [H'(x)]_{6 \times 2}^T \bar{Q}_m(x) \right) dx + \frac{t_e}{6} \left\{ \int_{0}^{t_e} \delta d_N e^{t \Omega} \right\}$$

(24)

The derived elastic stiffness, mass matrices, and load vector are intended for a one-dimensional, two-noded symmetric sandwich beam element with three degrees of freedom per node. The proposed element investigates the steady state dynamic analysis for coupled axial-bending response evaluated by using the exact shape functions developed in this formulation.

VERIFICATION AND EXAMPLES

The present finite beam element solution which governs the steady state extensional-bending coupling response of sandwich beams under various harmonic excitations, can be applied for the following analyses:

- To investigate the dynamic response for the sandwich beams under harmonic bending forces and moments at a certain exciting frequency $\Omega$.
- To capture the static response of the given sandwich beams under harmonic bending excitations by significantly lower exciting frequency $\Omega$ in contrast to the primary natural frequency $\omega_1$ of the sandwich beam (i.e., $\Omega \approx 0.01 \omega_1$, where $\omega_1$ is the fundamental natural frequency of the sandwich beam).
- To predict the natural frequencies and corresponding mode shapes of the sandwich beams under given forces.
To establish the validity and accuracy of the proposed finite sandwich beam element developed in this formulation, several examples are provided. These examples explore the coupled vibration of symmetric sandwich beams in axial-flexural bending, under different boundary conditions, and subjected to various harmonic forces and moments. The current finite element approach for sandwich beams relies on exact axial and flexural shape functions that precisely adhere to the solution of the governing coupled field equations. This treatment eliminates mesh discretization errors typically present in classical finite element methods employing polynomial interpolation shape functions. Consequently, it is observed that the results obtained from a single finite sandwich beam element for a clamped-free and two finite sandwich beam elements for other boundary conditions exactly coincide with the corresponding results derived from the exact closed-form solution developed in a previous study [9] up to five significant digits. Where, the present finite sandwich beam element has three degrees of freedom per node, i.e., 6 dof per element. Furthermore, numerical nodal displacement results obtained from the current finite sandwich beam element are compared with established finite element software Abaqus and exact solutions documented in the existing literature. Even though, the finite element developed in this study achieves the solution accuracy using one finite sandwich element for clamped-free and two sandwich elements for other boundary conditions (i.e., clamped-clamped, clamped-pinned and pinned-pinned) but in some cases for the sake of comparison more elements are required to show this accuracy.

In the finite element Abaqus solution, two different finite elements are performed to model the sandwich beam. The shell elements (S4R) are used to model the sandwich thin faces, while the sandwich core is modelled with solid brick element (C3D8R). The four-noded shell element (S4R) features six degrees of freedom per node, encompassing three translations and three rotations, effectively capturing shear deformation and cross-sectional distortional effects. In contrast, the eight-noded brick element (C3D8R) is a linear brick element with reduced integration, comprising three degrees of freedom per node (representing displacements in the x, y, and z directions), thus totaling 24 degrees of freedom per brick element.

**Example (1) – Verification of Results**

For verification purpose, a sandwich beam having identical faces and span $L = 0.9144m$ subjected to distributed transverse harmonic force $q_z(x,t) = 200e^{i\Omega t}N/m$ applied along the sandwich beam axis is illustrated in Figure 3. The geometric and material properties of the sandwich beam taken from [1] are used in the present example as: the face thicknesses $h_f = 0.4572mm$, core thickness $h_c = 12.70mm$, sandwich width $b = 25.40mm$, $E_fA_f = 31500N$, $E_fI_f = 1.362Nm^2$, $k_sG_cA_c = 1050N$, and $m_f = 0.001225kg/m$.

![Figure 3: A structural sandwich beam under harmonic excitation](image-url)
In Abaqus model, the thin faces of the sandwich are segmented into 91 S4R shell elements along the longitudinal axis of the beam, with two shell elements spanning the face width (i.e., Abaqus shell model for both faces comprise of 324 S4R elements), and the sandwich core is meshed by using 91 C3D8R eight-noded linear brick elements along the sandwich beam with two brick elements along the core height and core width, respectively, (i.e., Abaqus sandwich core used 324 C3D8R brick elements). The Abaqus sandwich beam model uses a total of 5800 dof to attain the required accuracy. In contrast, the finite element formulation devised in this study relies on exact shape functions and employs two finite sandwich beam elements, totaling 9 dof. Remarkably, the results obtained align exactly with those derived from the closed-form solution [9], accurate up to five significant digits. For the sake of comparison with other solutions, the present FE solution uses five sandwich beam elements to demonstrate the accuracy of the present element.

**Extracting of Natural Frequencies**

For a sandwich beam having clamped-free and clamped-clamped boundary conditions subjected to distributed transverse harmonic force \( q_z(x, t) = 200e^{i\Omega t} N/m \), the natural frequencies related to coupled axial-bending response are conducted from the steady state response analyses for an exciting frequency \( f \), changing from closely zero to 1800 Hz for clamped-free and 2200 Hz for clamped-pinned conditions. The nodal axial displacement, \( U_n \), transverse displacement, \( W_n \), and bending rotation, \( \phi_{xn} \), at the sandwich mid-span (i.e., \( x=L/2 \)) against the exciting frequency, \( \Omega \), are demonstrated for both cases (clamped-free and clamped-pinned) in Figure (4a,b,c) and Figure (5a,b,c), respectively. The natural frequencies are determined from the peaks observed in the displacement-frequency diagrams. Each peak signifies resonance and reveals the natural frequencies of the sandwich structural beam. Subsequently, the first five natural frequencies are identified from the peaks depicted in Figures (4a,4b,4c) and (5a,5b,5c) are provided in Tables 1 for sandwich beams with various boundary conditions (i.e., clamped-free CF, pinned-pinned PP, clamped-pinned CP and clamped-clamped CC). The natural frequencies of the sandwich beam, as extracted by the current finite element solution utilizing two beam elements, are compared with those reported by Lee [10]. It is clear that, the present FE results for natural frequencies demonstrate an excellent agreement.

<table>
<thead>
<tr>
<th>Boundary condition</th>
<th>Reference</th>
<th>( f_1 )</th>
<th>( f_2 )</th>
<th>( f_3 )</th>
<th>( f_4 )</th>
<th>( f_5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>CF</td>
<td>Lee [10]</td>
<td>31.459</td>
<td>193.71</td>
<td>529.20</td>
<td>1006.4</td>
<td>1612.8</td>
</tr>
<tr>
<td></td>
<td>Present FE</td>
<td>31.449</td>
<td>193.70</td>
<td>529.19</td>
<td>1006.2</td>
<td>1612.5</td>
</tr>
<tr>
<td>PP</td>
<td>Lee [10]</td>
<td>166.18</td>
<td>445.12</td>
<td>852.16</td>
<td>1381.5</td>
<td>2030.7</td>
</tr>
<tr>
<td></td>
<td>Present FE</td>
<td>166.23</td>
<td>445.20</td>
<td>852.39</td>
<td>1381.6</td>
<td>2031.6</td>
</tr>
<tr>
<td>CP</td>
<td>Lee [10]</td>
<td>179.65</td>
<td>481.22</td>
<td>918.52</td>
<td>1481.7</td>
<td>2165.4</td>
</tr>
<tr>
<td></td>
<td>Present FE</td>
<td>179.70</td>
<td>481.41</td>
<td>918.89</td>
<td>1482.3</td>
<td>2166.3</td>
</tr>
<tr>
<td>CC</td>
<td>Lee [10]</td>
<td>194.91</td>
<td>521.79</td>
<td>991.81</td>
<td>1590.1</td>
<td>2308.9</td>
</tr>
<tr>
<td></td>
<td>Present FE</td>
<td>194.86</td>
<td>521.72</td>
<td>991.60</td>
<td>1589.9</td>
<td>2308.5</td>
</tr>
</tbody>
</table>
Figures (4d,e,f) and (5d,e,f) show the first five steady state mode shapes for normalized axial displacement \(U_n/U_{n\text{max}}\), transverse displacements \(W_n/W_{n\text{max}}\), and bending rotation \(\Phi_{x_n}/\Phi_{x_{n\text{max}}}\) of sandwich beams having clamped-free and clamped-pinned boundary conditions, respectively. For the sake of comparison, five elements are used for the present finite element solution. In other words, both the solution described in paper [9] and the finite element solution developed in the current work are depicted together on the same diagrams, demonstrating a strong match. In general, the finite element approach effectively captures the natural frequencies and modes of the sandwich beams.

Figure 4: Natural frequencies and modes of clamped-free sandwich beam
An Efficient Finite Element for Vibration Analysis of Symmetric Sandwich Beams Subjected to Harmonic Bending Excitations

Figure 5: Natural frequencies and modes for clamped-pinned sandwich beam

Example (2): Static and dynamic responses

To validate and confirm the accuracy of developed finite element solution to approach the quasi-static and dynamic responses, the present results conducted in this example are compared with Abaqus finite element model. A clamped-clamped sandwich beam under uniformly distributed transverse force $q_z(x, t) = 100e^{int}N/m$, is considered as demonstrated in Figure 6. The sandwich beam has the same length, geometric and material properties as in example 1, except that the sandwich width is taken as $b = 25.4\text{mm}$. It is required to compare the static and steady state dynamic responses obtained from the present finite element solution with the
Abaqus finite element model. The behavior of the sandwich beam under the given force is analyzed in two scenarios: Firstly, its response at a very low exciting frequency, which is close to its first natural frequency, is observed to capture its quasi-static behavior. Secondly, its steady-state dynamic response is computed using an exciting frequency $\Omega = 300\, \text{rad/SEC}$.

Three approaches are employed to address the sandwich beam problem. The first approach relies on a closed-form solution presented in a prior study [9]. The second approach utilizes the current finite element formulation, wherein the sandwich beam is divided into just five beam elements along its span, resulting in a model with only 18 degrees of freedom. This outcome is a consequence of employing exact shape functions in the finite element formulation, thereby minimizing mesh discretization errors. The third approach involves an Abaqus shell element model, dividing the sandwich beam into 90 shell S4R elements longitudinally and two elements each along the width of the top and bottom faces, respectively, while the sandwich core uses 100 C3D8R eight-noded linear brick elements along the sandwich axis, two brick elements along the width and height of the sandwich core, respectively. Therefore, the sandwich beam model consists of 360 shell S4R elements and 360 C3D8R brick elements, which leads to approximately 11,600 dof were necessary to mitigate discretization errors and achieve the desired level of accuracy.

Quasi-static Response

The nodal axial $U_n$, transverse displacement $W_n$ and related bending rotation $\Phi_{x_n}$ (for $n = 1, 2, 3, \ldots, 6$) static results obtained through the current finite element approach are contrasted with the exact solution provided in [9] and Abaqus finite element model. Figure (7a,b,c) illustrates that the nodal degrees of freedom outcomes predicted by the current finite element solution (which neglects shear deformation effects in the sandwich faces) using five beam elements totaling 18 degrees of freedom align closely with those derived from the exact closed-form solution [9]. Additionally, they closely match the corresponding results obtained from the Abaqus shell solution, utilizing 360 S4R shell elements on the faces (which account for shear deformation effects in the sandwich faces) and 360 C3D8R linear brick elements in the sandwich core having approximately a total of 11,600 dof in the sandwich beam model. This indicates that the shear deformation effects captured in Abaqus shell model have little influence on the static responses for the present sandwich beams. This leads to conclude that, as a general remarks, the present finite element solution is successful in capturing the static response of the given sandwich beam system.
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Steady State Dynamic Response

The dynamic response nodal results of axial displacement $U_n$, transverse displacement $W_n$, and related bending rotation $\Phi_{xn}$ versus the sandwich beam coordinate axis are exhibited in Figure (7d,e,f), respectively, for sandwich beam under distributed harmonic force $q(x,t) = 200e^{i\Omega t} N/m$ with exciting frequency $\Omega = 300 rad/sec$.

The steady-state dynamic findings derived from the current finite element method precisely match the closed-form solution outlined in [9] but exhibit slight disparities compared to those obtained from the Abaqus finite element model. Once again, these differences stem from the inclusion of shear deformation effects in the Abaqus finite shell element solution, which are not included in the current finite element formulation. Notably, the present finite element formulation, employing only five beam elements (for the sake of comparison) yields results closely resembling those from the Abaqus finite element solution employing 720 elements, while requiring significantly less computational resources and modeling effort. This outcome is inherent to the present finite element’s utilization of shape functions that precisely fulfill the homogeneous solution of the coupled field equations, thereby eliminating discretization errors typically encountered in finite element formulations.
Example (3): Sandwich span and core Thickness effects

To demonstrate how the thickness of the core and the beam span affect its natural frequencies, as well as its responses in static and dynamic states, a three-layered sandwich beam with identical faces is considered. This sandwich beam is clamped at one end and free at the other, and it experiences a concentrated harmonic force at its end $F_z(L, t) = 400e^{i\Omega t} N$, as illustrated in Figure 8. The sandwich beam of width 40mm has equal thicknesses for its top and bottom faces, which are both 2.0mm. The material properties for the faces are: $E_f = 70GPa$, $\rho_f = 2700kg/m^3$, and the core layer properties are: $G_c = 80MPa$, and $\rho_c = 100kg/m^3$.

The aim is to explore how varying the core thickness and beam span influence its natural frequencies, as well as its responses under quasi-static and steady-state dynamic conditions.

Sandwich Span Effect on Natural Frequencies

Under the given end harmonic force, the steady-state dynamic analyses are carried out to anticipate the natural frequencies of the given sandwich beams. The initial five natural frequencies (measured in Hz) obtained from the dynamic responses, as illustrated in Figure 9, are determined using the current finite element approach, employing a single beam element for core thickness $h_c = 50mm$ and various sandwich spans ($L = 1.0, 1.5, 2.0, 2.5$ and $3.0m$). It is noted that with increasing sandwich span, the natural frequency decreases across different steady-state mode shapes. This pattern may be attributed to the inverse relationship between natural frequency and sandwich beam span.
Effect of Core Thickness on Static and Dynamic Responses

The static and steady state dynamic analyses of the sandwich clamped-free beam of length \( L = 2.0m \) under the given concentrated harmonic force \( F_z(L, t) = 200e^{i\Omega t} N \) with exciting frequencies \( \Omega \approx 0.01\omega_1 \) and \( \Omega = 200\text{rad/sec} \) are investigated, respectively. The static and dynamic results for nodal axial displacement \( U_n \), transverse displacement \( W_n \), and bending rotation \( \Phi_{xn} \) are illustrated in Figures (10a,b,c) and (10d,e,f), respectively. Although, the developed finite element results obtained based on a single sandwich beam element for the case of clamped-free and two finite sandwich elements for other boundary conditions, but five beam elements are employed for the comparison purpose in order to demonstrate the accuracy of the present element with the corresponding results computed from the exact solution given in [9]. The nodal degrees of freedom results obtained from both solutions are graphed on identical diagrams to facilitate comparison. The results obtained through the present finite element formulation are found to perfectly align with those derived from the exact closed-form solution. The observation of these diagrams reveals that, the static and dynamic results obtained from the current formulation demonstrated outstanding consistency with those obtained from the previous closed form solution. Figures (10a,b,c) and (10d,e,f) investigate the sandwich core thickness effect on the quasi-static and dynamic responses. Plots of the maximum nodal axial displacement \( U_n \), transverse displacement \( W_n \), and corresponding bending rotation \( \Phi_{xn} \) (where \( n=6 \)) are presented for symmetric sandwich beams with varying core thicknesses. It is seen that as the core thickness increases, the nodal axial and transverse displacements decrease. This trend holds true for static responses. However, in dynamic responses, the effects are reversed. Generally, increasing the core thickness enhances the stiffness of the sandwich beam in static response scenarios, whereas the opposite holds true for dynamic responses.
Figure 10: Static and dynamic responses of sandwich clamped-free beam of span $L=2.0m$ under end transverse force with $\Omega = 200rad/sec$

**Effects of Sandwich Span on Static Response**

To investigate the effect of sandwich span on the quasi-static response for the symmetric sandwich beam having different core thicknesses $h_2 = 20, 40, 60, 80mm$, the maximum nodal axial displacement $U_n$, transverse displacement $W_n$ and related rotation $\Phi_{xn}$ (where $n = 6$) at sandwich beam free end are plotted for different core thicknesses as in Figure (11a,b,c), respectively. It is obvious that, as the core thickness increase from 20mm to 80mm, all the maximum nodal degrees of freedom are decreased. It is also seen that all three nodal static responses are increased when the sandwich length is increased from (from 0.5m to 3.0m).
indicates that quasi-static responses increase with an increase in sandwich beam length but they decline with increasing core thickness.

Dynamic response analysis

The nodal steady state axial displacement $U_n$, transverse displacement $W_n$ and related bending rotation $\Phi_{xn}$ (for $n = 1,2,3,\ldots,6$) dynamic responses of sandwich beam under harmonic force are illustrated in Figures (13a,b,c), respectively, for different values of frequency ratios $\Omega_i/\omega_1$ (for $i = 1,2,3,4$), i.e., applied load frequencies $\Omega_i$ to the fundamental natural frequency $\omega_1$ of the specified sandwich beam having length $L = 2.0m$ and core thickness $h_c = 60mm$. Figure (12a,b,c) presents the nodal axial displacement, transverse displacement and related rotation responses of the sandwich beam under harmonic force $P(x,t) = 400e^{i\Omega t}N$ for four values of exciting frequency $\Omega_2 = 0.80\omega_1$, $\Omega_3 = 6.80\omega_1$, $\Omega_3 = 12.80\omega_1$, and $\Omega_4 = 18.80\omega_1$, where $f_1 = 30.01Hz$ is the fundamental natural frequency of the beam. For the sake of comparison, the nodal dynamic results of finite element developed in this study used five beam elements and exact closed form solution investigated in previous work [9] are plotted on the same diagrams. Again, it is observed that, the nodal displacements results derived from the current finite element approach are exactly matched with those computed based on the closed form solution.

![Graphs showing dynamic response analysis](image-url)
SUMMARY AND CONCLUSION

- A highly effective finite beam element is formulated for analysing symmetric sandwich beams subjected to different harmonic bending forces.
- The novel two-noded sandwich beam element utilizes shape functions that exactly meet the requirements of the homogeneous solution for the coupled axial-bending dynamic equations.
- The sandwich beam element effectively eliminates the discretization errors often encountered in other interpolation schemes, consistently delivering exceptional results while utilizing a considerably smaller number of degrees of freedom.
- The current finite element formulation adeptly captures both the quasi-static and steady-state dynamic responses of sandwich beams subjected to different harmonic bending forces.
- It also has the ability to extract eigen-frequencies and eigen-modes from the steady-state dynamic response of sandwich beams.
- The finite element developed in this formulation demonstrates remarkable consistency with Abaqus finite element results, all while requiring significantly fewer computational resources and modelling efforts.

REFERENCES


An Efficient Finite Element for Vibration Analysis of Symmetric Sandwich Beams Subjected to Harmonic Bending Excitations


عنصر محدود فعال لتحليل اهتزاز عتبات الساندوتش المتماثلة المعرضة لتأثير الانحناء التوافقي

حسن المهدي النجار و محمد علي الحجاجي
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ملخص البحث
تم تطوير عنصر محدود فعال لعراضات الساندوبش من أجل التحليل الدينيميكي للانحناء-المحوري المقترن لعراضات الساندوبش المتماثلة المعرضة لقوى الانحناء التوافقية المختلفة. تم استخدام صيغة هاملتون التغابنية لتشكل معادلات الحركة الحاكمة، والتي تم حلها لإيجاد الحل الدقيق للإفاعات الدينيمية للحالة المستقرة. تم بعد ذلك تطوير مجموعة من دوال الأشكال الدقيقة بناء على الحل الدقيق لمعادلات الإفاعات المزدوجة وتم استخدامها لصياغة عنصر محدود الدقيق للعجارة. يحتوي العنصر الجديد على عقدتين مع ست درجات من الحركة لكل عنصر ويلتقط بنجاح استجابات الحالة الثابتة والانحناء المحوري المقترن لعراضات الساندوبش المتماثلة تحت قوى الانحناء التوافقية. من أجل التحقق من دقة ونفعية عنصر العجارة الحالي، تم إجراء مقارنات مع حلول العناصر المحدودة Abaqus الأخرى. تظهر النتائج أن العنصر المحدود الجديد خالٍ من أخطاء التقسيم الناتجة من استخدام الدوال التقربية الأخرى ويتنج عنه اتفاق معتمد مع تلك المستندة إلى حلول العناصر المحدودة الأخرى بجزء صغير من التكلفة الحسابية والنموذج المعنية.

الكلمات المفتاحية: عنصر محدود فعال؛ دوال الشكل الدقيقة؛ القوى التوافقية.